Research Article

New Traveling Wave Solutions and Interesting Bifurcation Phenomena of Generalized KdV-mKdV-Like Equation

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Using the bifurcation method of dynamical systems, we investigate the nonlinear waves and their limit properties for the generalized KdV-mKdV-like equation. We obtain the following results: (i) three types of new explicit expressions of nonlinear waves are obtained. (ii) Under different parameter conditions, we point out these expressions represent different waves, such as the solitary waves, the 1-blow-up waves, and the 2-blow-up waves. (iii) We revealed a kind of new interesting bifurcation phenomenon. The phenomenon is that the 1-blow-up waves can be bifurcated from 2-blow-up waves. Also, we gain other interesting bifurcation phenomena. We also show that our expressions include existing results.

1. Introduction

Most relationships in nature and human society are intrinsically nonlinear rather than linear in nature, so many phenomena in nature and human society can be described by nonlinear equations, such as automatic control, meteorology, engineering calculation, engineering budget, economy, and finance [1, 2]. Nowadays, many scientists are very interested in nonlinear equations and their solutions and have done a lot of related work [3–5].

In the paper, we consider the generalized KdV-mKdV-like equation [6, 7].

$$u_t + \left(\alpha + \beta u^p + \gamma u^{2p} \right) u_x + u_{xxx} = 0, \tag{1}$$

where $p > 0$, α, β, $\gamma \neq 0$ are real constants. By using appropriate parameters, the generalized KdV-mKdV-like equation becomes the classical KdV equation [8–11], the mKdV equation [12–16], the KdV-like equation [17–20], and the generalized mKdV equation [21].

Up to now, many authors have been interested in the study of the many forms of KdV-like equations [22–25], and there are several explicit solutions results of the generalized KdV-mKdV-like equation based on the significant physical background. For example, Li and Wang [6] gave the following traveling wave solution:

$$u_w(\xi) = \left[\frac{-2(p+1)(p+2)(2p+1)\,\beta}{(2p+1)\beta^2 (p\,\xi)^2 + (p+1)(p+2)^2\,\gamma} \right]^{1/p}, \tag{2}$$

where $\beta < 0$, $\gamma > 0$, $\xi = x - \alpha\, t$.

In recent years, the bifurcation method of dynamical systems has been widely used in investigating the nonlinear partial differential equations, for instance [26–29].

In this paper, we study the nonlinear wave solutions and the bifurcation phenomena for Eq. (1). First, we obtain three types of explicit waves which represent the solitary waves, the 1-blow-up waves, and the 2-blow-up waves. Second, we reveal the new bifurcation phenomena which are introduced in the abstract above. Furthermore, we obtain other interesting bifurcation phenomena. The first phenomenon is that the 1-blow-up waves can be bifurcated from the solitary waves. The second phenomenon is that the trivial waves can be bifurcated from the solitary waves.

This paper is organized as follows. In Section 2, we give some notations and state our main results. Our main
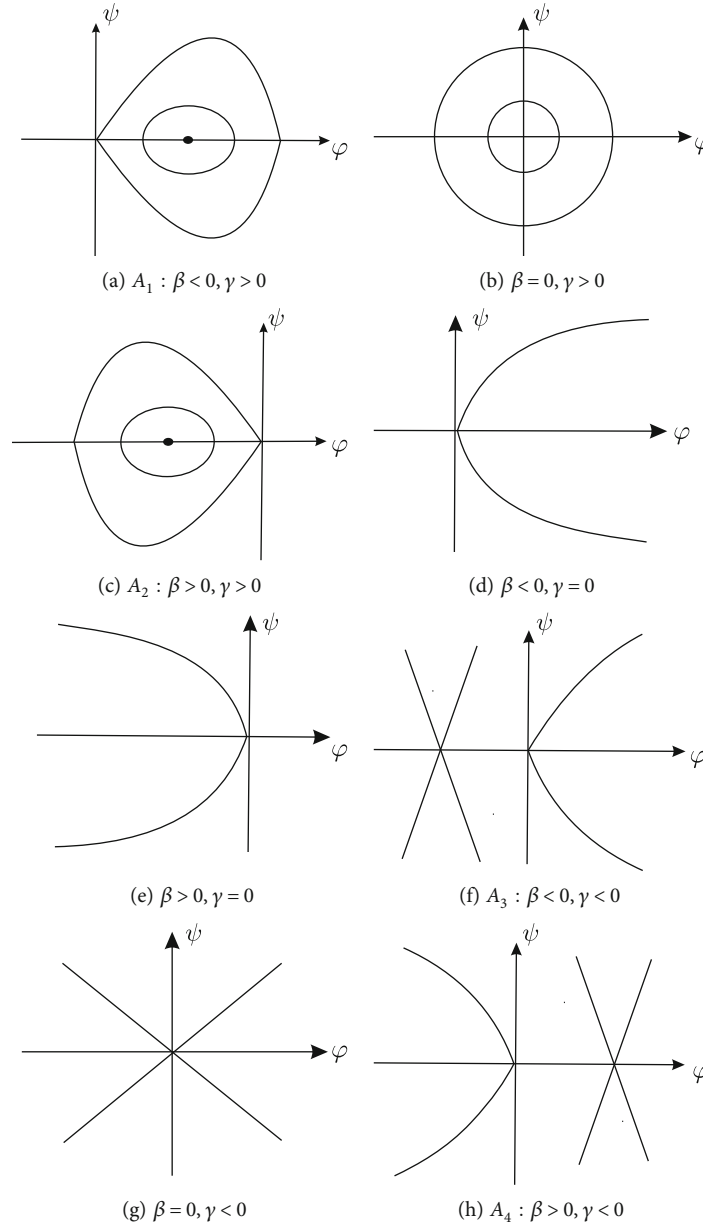
(a) $A_1 : \beta < 0, \gamma > 0$

(b) $\beta = 0, \gamma > 0$

(c) $A_2 : \beta > 0, \gamma > 0$

(d) $\beta < 0, \gamma = 0$

(e) $\beta > 0, \gamma = 0$

(f) $A_3 : \beta < 0, \gamma < 0$

(g) $\beta = 0, \gamma < 0$

(h) $A_4 : \beta > 0, \gamma < 0$

FIGURE 1: The phase portraits of the system (12).

derivations are listed in Section 3. A brief conclusion is given in Section 4.

2. Our Main Results

In this paper, p is odd and the situation of even is similar to study. In this section, we state our main results. In order to state these results conveniently, we give some notations which will be used in the latter statement and the derivations.

The zones A_j $(j = 1, 2, 3, 4)$ are given in Figure 1, and κ is an arbitrary real constant. In this article, we only consider the case $\alpha - c = 0$. For other cases, due to the complexity, we will investigate them in our future works.

Proposition 1. If $\alpha - c = 0$, then, the explicit solutions are

$$u_1(\xi) = \left(\frac{-2(p+1)(p+2)(2p+1)\,\beta}{(2p+1)\beta^2(p\,\xi + \kappa)^2 + (p+1)(p+2)^2\,\gamma} \right)^{1/p}, \quad (3)$$

and

$$u_2(\xi) = \left(\frac{1}{p\,\sqrt{-\gamma/(p+1)(p+2)}\xi + \kappa} \right)^{1/p}, \quad (4)$$

(a) $\gamma = 10^{-1}$
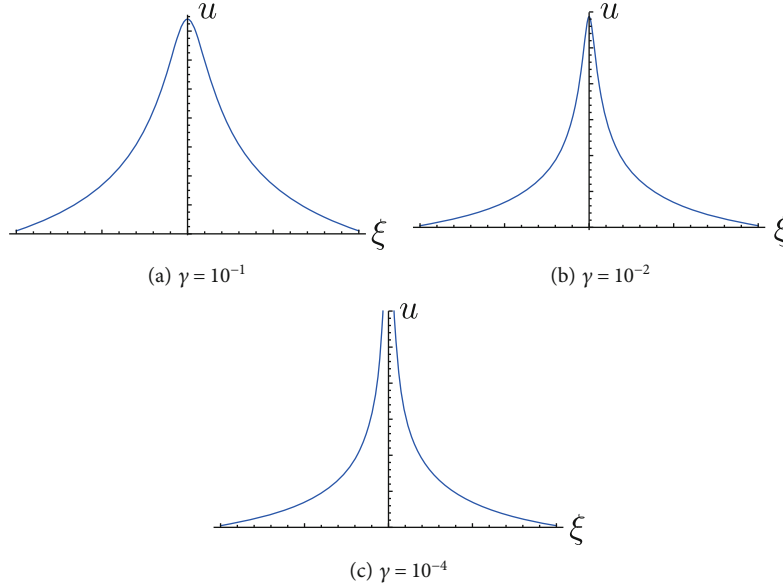

(b) $\gamma = 10^{-2}$

(c) $\gamma = 10^{-4}$

FIGURE 2: The varying figures of the example of $u = u_1(\xi)$ when $\kappa = 0, p = 5, \alpha - c = 0, \beta = -3$, and $\gamma \longrightarrow 0 + 0$.

when $\kappa = 0$, u_1 becomes

$$u_1^0(\xi) = \left(\frac{-2(p+1)(p+2)(2p+1)\beta}{(2p+1)(\beta p \xi)^2 + (p+1)(p+2)^2 \gamma} \right)^{1/p}, \quad (5)$$

After selecting the appropriate parameters, u_1^0 is equivalent to u_w.

when $\gamma = 0$, u_1 becomes

$$u_1^1(\xi) = \left(\frac{-2(p+1)(p+2)}{\beta (p \xi + \kappa)^2} \right)^{1/p}, \quad (6)$$

(i) If $(\beta, \gamma) \in A_1$ or A_2, then, u_1 is symmetric solitary wave (the example is given in Figure 2(a) or Figure 3(a)). Specially, when $\gamma \longrightarrow 0 + 0$, then, the symmetric solitary wave u_1 becomes single-side 1-blow-up wave u_1^1 (the example is given in Figure 2(c)), and for the varying process of the example, see Figure 2. When $\beta \longrightarrow 0 \pm 0$, then, the symmetric solitary wave u_1 becomes the trivial wave (the example is given in Figure 3(c)), and for the varying process of the example, see Figure 3

(ii) If $(\beta, \gamma) \in A_3$ or A_4, then, u_1 is 2-blow-up solitary wave (the example is given in Figure 4(a)). Specially, when $\gamma \longrightarrow 0 - 0$, then, the 2-blow-up wave u_1 becomes the single-side 1-blow-up wave u_1^1 (the example is given in Figure 4(c)), and for the varying process of the example, see Figure 4

(iii) If $\beta = 0, \gamma < 0$, then, u_2 is 1-blow-up solitary wave

3. The Derivation of Main Results

To derive our results, we give some preliminaries in this section. For simplicity of the derived expression, we use the following notation

$$A = \frac{\gamma}{2(p+1)(2p+1)}, \quad (7)$$

$$B = \frac{\beta}{(p+1)(p+2)}, \quad (8)$$

$$C = \frac{\alpha - c}{2}. \quad (9)$$

then we derive our main results.

3.1. The Derivations to Proposition 1. For given constant c and $c - \alpha = 0$, substituting $u = \varphi(\xi)$ with $\xi = x - ct$ into Eq.(1), it follows that

$$\beta \varphi^p \varphi' + \gamma \varphi^{2p} \varphi' + \varphi''' = 0. \quad (10)$$

Integrating (10) once and letting the integral constant be zero, we get

$$\frac{\beta}{p+1} \varphi^{p+1} + \frac{\gamma}{2p+1} \varphi^{2p+1} + \varphi'' = 0. \quad (11)$$

Letting $\psi = \varphi'$, we obtain a planar system

$$\begin{cases} \dfrac{d\varphi}{d\xi} = \psi, \\ \dfrac{d\psi}{d\xi} = -\dfrac{\gamma}{2p+1} \varphi^{2p+1} - \dfrac{\beta}{p+1} \varphi^{p+1}, \end{cases} \quad (12)$$
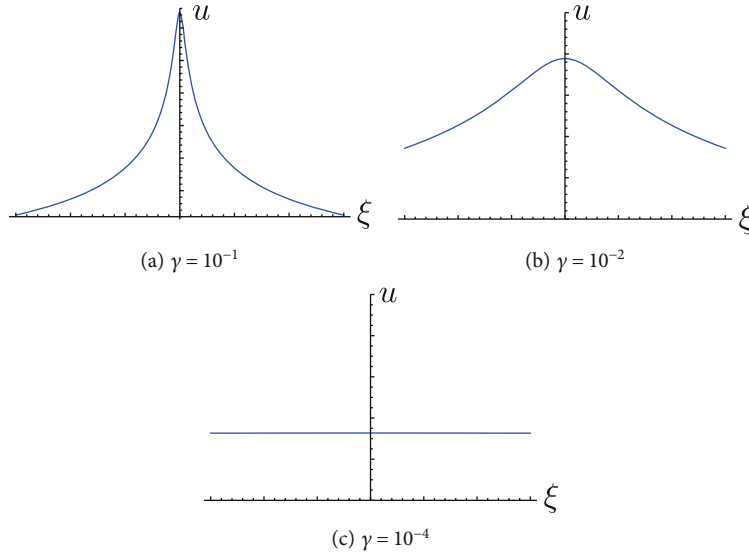
(a) $\gamma = 10^{-1}$

(b) $\gamma = 10^{-2}$

(c) $\gamma = 10^{-4}$

FIGURE 3: The varying figures of the example of $u = u_1(\xi)$ when $\kappa = 0, p = 9, \alpha - c = 0, \gamma = 1$, and $\beta \longrightarrow 0 - 0$.

(a) $\gamma = 10^{-1}$

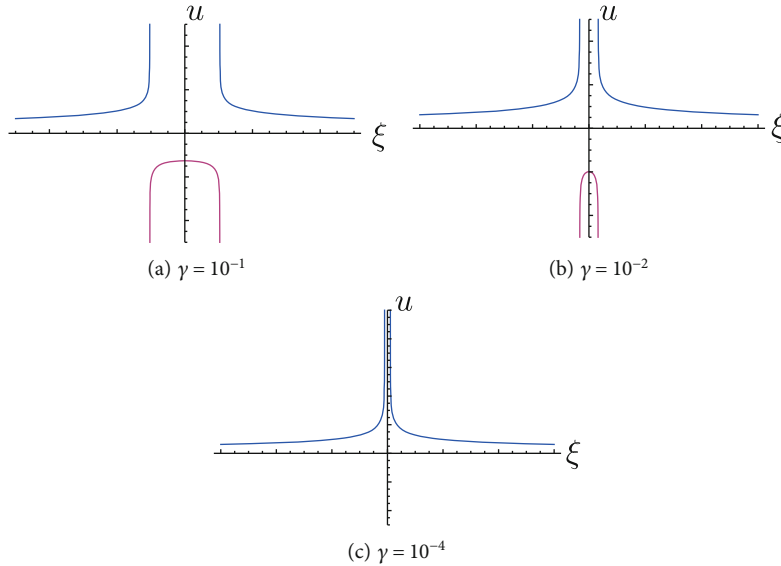(b) $\gamma = 10^{-2}$

(c) $\gamma = 10^{-4}$

FIGURE 4: The varying figures of the example of $u = u_1(\xi)$ when $\kappa = 0, p = 9, \alpha - c = 0, \gamma = 1$, and $\beta \longrightarrow 0 - 0$.

with the first integral

$$H(\varphi, \psi) = \frac{1}{2}\psi^2 + \frac{\gamma}{2(p+1)(2p+1)}\varphi^{2p+2} + \frac{\beta}{(p+1)(p+2)}\varphi^{p+2} = h, \tag{13}$$

where h is the integral constant. According to the qualitative theory, we obtain the bifurcation phase portraits of system (12) as Figure 1. By means of the bifurcation phase portraits, we can derive Proposition 1.

In the first integral (13), letting $h = H(0,0)$, we obtain

$$\psi^2 = -2\varphi^2\left(A\varphi^{2p} + B\varphi^p\right), \tag{14}$$

Substituting (14) into the first equation of (12) and inte-

grating it, we get

$$\int_l^\varphi \frac{ds}{s\sqrt{As^{2p} + Bs^p}} = |\xi|, \tag{15}$$

where l is an arbitrary constant or $\pm\infty$.

When $\beta \neq 0$ and completing the integral above and solving the equation for φ, it will follow that

$$\varphi = \left(\frac{-2(p+1)(p+2)(2p+1)\beta}{(2p+1)\beta^2(p\xi + \kappa)^2 + (p+1)(p+2)^2\gamma}\right)^{1/p}, \tag{16}$$

and letting $\kappa = 0$, we can obtain (5) from (3). Similarly, when $\beta = 0$ and completing the integral above and solving the equation for φ, we gain (4). Therefore, we have completed the derivations for Proposition 1.

4. Conclusion

In this paper, we have investigated the explicit expressions of the nonlinear waves and their bifurcations in Eq. (1).

First, we obtained three types of new expressions. And they represent different waves, such as the solitary waves, the 1-blow-up waves, and the 2-blow-up waves.

Second, we revealed three kinds of bifurcation phenomena which include a new bifurcation phenomena. The first phenomenon which is new bifurcation phenomenon is that 1-blow-up waves can be bifurcated from 2-blow-up waves. The second phenomenon is that the trivial waves can be bifurcated from the solitary waves. The third phenomenon is that the 1-blow-up waves can be bifurcated from the solitary waves.

Third, we showed that a previous result is our special case, that is, u_w is included in u_1^0.

Furthermore, the bifurcation method of dynamical systems can be used to find the new traveling solutions and bifurcations of many nonlinear equations such as the extended quantum Zakharov-Kuznetsov equation [37], the Fujimoto-Watanabe equation [38], and b-family-like equation [39]. We will continue to use the bifurcation method of dynamical systems to study other important nonlinear equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgments

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