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Some Inequalities on Generalized $p - k$ Gamma and Beta Functions

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this work, we firstly introduce the integral representation of the generalized $p - k$ beta function and then employ well-known Chebychev's and Hölder's inequalities to obtain some inequalities on generalized $p - k$ gamma and beta functions. Also, we show the logarithmic convexity properties of these functions.

Keywords: Inequality; generalized $p - k$ gamma function; generalized $p - k$ beta function; convexity.

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1 Introduction

One of the fundamental areas of mathematical analysis is inequalities. The theory of integral inequalities has been used in many scientific fields such as physics, statistics and probability (see for example [1, 2, 3] and references therein). Since special functions such as gamma function also play a major role in many subjects, many inequalities involving gamma function and its extensions are presented, [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

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Nowadays, Gehlot introduces the two-parameter gamma function ${}_p\Gamma_k(x)$ in [8] and after in [11], a new two-parameter gamma function ${}_p^a\Gamma_k(x)$ named as generalized $p-k$ gamma function. The main objective is to give some inequalities on generalized $p-k$ gamma and beta functions.

This paper is organized as follows: In Section 2, we give some notations and preliminaries for the convenience of the reader. Also, we give the definition and some properties of the generalized $p-k$ beta function. In Section 3 and 4, we deduce some inequalities and convexity properties of ${}_p^a\Gamma_k(x)$ and ${}_p^aB_k(x)$ functions via Chebychev's and Hölder's integral inequalities.

2 Notations and Preliminaries

The integral representations of the generalized k-gamma and k-beta functions are defined as

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, k>0,\, x\in\mathbb{C}-k\mathbb{Z}^- \tag{2.1}$$

and

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt \tag{2.2}$$

respectively in [5].

The k-gamma function Γ_k is one of the generalizations of the classical gamma function with one parameter and in order to obtain some results on the function such as its limit expression and infinite product expression, one needs the following the Pochhammer k-symbol $(x)_{n,k}$:

$$(x)_{n,k} = x(x+k)\dots(x+(n-1)k), \quad n\in\mathbb{N},\, k>0.$$

For $x\in\mathbb{C}$, $Re(x)>0$, $k,p\in\mathbb{R}^+-\{0\}$, $n\in\mathbb{N}$ and $a\in(1,\infty)$ the generalized $p-k$ Pochhammer symbol ${}_p^a(x)_{n,k}$ is given by

$${}_p^a(x)_{n,k} = \left(\frac{xp}{k\log a}\right)\left(\frac{xp}{k\log a}+\frac{p}{\log a}\right)\left(\frac{xp}{k\log a}+\frac{2p}{\log a}\right)\dots\left(\frac{xp}{k\log a}+\frac{(n-1)p}{\log a}\right) \tag{2.3}$$

and the generalized two-parameter Gamma function ${}_p^a\Gamma_k$ is given by

$${}_p^a\Gamma_k(x) = \frac{1}{k}\lim_{n\to\infty}\frac{p^{n+1}n!}{(\log a)^{n+1}}\frac{1}{{}_p^a(x)_{n,k}}\left(\frac{np}{\log a}\right)^{\frac{x}{k}-1}. \tag{2.4}$$

The integral representation of the function ${}_p^a\Gamma_k$ is defined by

$${}_p^a\Gamma_k(x) = \int_0^\infty a^{-\frac{t^k}{p}}t^{x-1}dt. \tag{2.5}$$

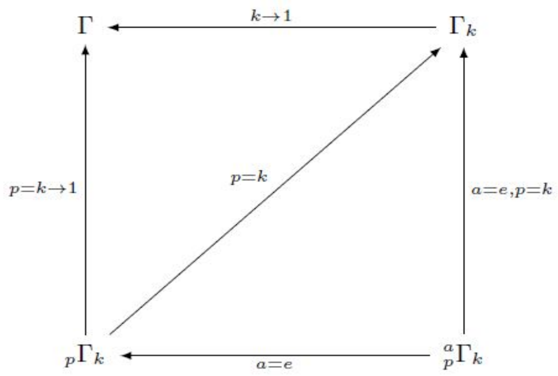
The relation between generalized $p-k$ gamma function, k-gamma function and classical gamma function is given in [11] such that

$${}_p^a\Gamma_k(x) = \frac{{}_p\Gamma_k(x)}{(\log a)^{\frac{x}{k}}} = \left(\frac{p}{k\log a}\right)^{\frac{x}{k}}\Gamma_k(x) = \frac{1}{k}\left(\frac{p}{\log a}\right)^{\frac{x}{k}}\Gamma\left(\frac{x}{k}\right). \tag{2.6}$$

The integral representation (2.5) gives that the generalized two-parameter Gamma function ${}_p^a\Gamma_k$ is an infinitely differentiable function on $(0,\infty)$ and the nth derivative of the generalized $p-k$ gamma function is

$${}_p^a\Gamma_k^{(n)}(x) = \int_0^\infty t^{x-1}\ln^n t\, a^{-\frac{t^k}{p}}dt, \quad n\in\mathbb{N}.$$

Note that, generalized $p-k$ gamma function is a deformation of the classical gamma function and satisfies the following commutative diagram:



where ${}_p\Gamma_k$ is defined as two-parameter gamma function given in [8].

Also in [11], the author discussed the following properties:

$$
\begin{aligned}
{}^a_p\Gamma_k(x+k) &= \frac{xp}{k\log a}{}^a_p\Gamma_k(x), && (2.7)\\
{}^a_p(x)_{n,k} &= \frac{{}^a_p\Gamma_k(x+nk)}{{}^a_p\Gamma_k(x)}, && (2.8)\\
{}^a_p\Gamma_k(x) &= \frac{p}{k\log a}. && (2.9)
\end{aligned}
$$

Since the classical beta function, an useful function of two variables can be evaluated in terms of the Euler's classical Gamma function as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad Re(x), Re(y) > 0$$

it is natural to define the generalized $p-k$ beta function ${}^a_pB_k(x,y)$ as follows:

Definition 2.1. For $x, y \in \mathbb{C}$; $k, p \in \mathbb{R}^+ - \{0\}$ and $Re(x), Re(y) > 0$, $n \in \mathbb{N}$, $a \in (1,\infty)$, the generalized $p-k$ beta function is defined by

$$
{}^a_pB_k(x,y) = \frac{{}^a_p\Gamma_k(x){}^a_p\Gamma_k(y)}{{}^a_p\Gamma_k(x+y)}. \tag{2.10}
$$

Proposition 2.1. *The function ${}^a_pB_k$ satisfies the following identities:*

$$
{}^a_pB_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}, \tag{2.11}
$$

$$
{}^a_pB_k(x,y) = B_k(x,y) = \frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right), \tag{2.12}
$$

$$
{}^a_pB_k(x,y) = {}^a_pB_k(y,x), \tag{2.13}
$$

for $x, y \in \mathbb{C}$; $k, p \in \mathbb{R}^+ - \{0\}$ and $Re(x), Re(y) > 0$, $n \in \mathbb{N}$, $a \in (1,\infty)$.

Proof. The results follow by using the equations (2.6) and (2.10). $\square$

3 Inequalities via Chebychev's One and Applications

In this section, we prove some applications to Chebychev's integral inequalities for generalized $p-k$ gamma and generalized $p-k$ beta functions. Firstly, we will recall the results which are known in the literature as Chebychev's integral inequalities for synchronous (asynchronous) mappings.

Lemma 3.1. *[6] Let $f, g, h : I \subset \mathbb{R} \to \mathbb{R}$ be so that $h(x) \geq 0$ for $x \in I$ and h, hfg, hf and hg are integrable on I. If f, g are synchronous (asynchronous) on I, i.e.*

$$(f(x) - f(y))(g(x) - g(y)) \geq (\leq) 0 \text{ for all } x,\, y \in I,$$

then we have the inequality

$$\int_I h(x)dx \int_I h(x)f(x)g(x)dx \geq (\leq) \int_I h(x)f(x)dx \int_I h(x)g(x)dx. \tag{3.1}$$

Theorem 3.2. *Let m, n, x, y be positive numbers with the property that*

$$(m-x)(n-y) \geq (\leq) 0$$

and i be a non-negative integer. Then

$${}_p^a\Gamma_k^{(2i)}(x+n)\, {}_p^a\Gamma_k^{(2i)}(y+m) \geq (\leq) {}_p^a\Gamma_k^{(2i)}(x+y)\, {}_p^a\Gamma_k^{(2i)}(m+n) \tag{3.2}$$

for $p, k > 0$ and $a \in (1, \infty)$.

Proof. Let us define $f, g, h : [0, \infty] \to [0, \infty)$ by $f(t) = t^{y-n}$, $g(t) = t^{m-x}$ and $h(t) = t^{x+n-1} \ln^{2i} t a^{-\frac{t^k}{p}}$. Differentiating the functions f and g yields that

$$f'(t) = (y-n)\, t^{y-n-1} \text{ and } g'(t) = (m-x)\, t^{m-x-1}$$

for $t \in (0, \infty)$. If $(m-x)(y-n) > 0$ then f and g have the same monotonicity i.e. they are synchronous and if $(m-x)(y-n) < 0$ then f and g have opposite monotonicity i.e. they are asynchronous. Also since i is a non-negative integer, $h(t) \geq 0$ on $[0, \infty)$. Hence using Chebychev's inequality (3.1) for the functions f, g and h lead us to

$$\int_0^\infty t^{x+n-1} \ln^{2i} t a^{-\frac{t^k}{p}} dt \int_0^\infty t^{y+m-1} \ln^{2i} t a^{-\frac{t^k}{p}} dt$$

$$\geq (\leq) \int_0^\infty t^{x+y-1} \ln^{2i} t a^{-\frac{t^k}{p}} dt \int_0^\infty t^{m+n-1} \ln^{2i} t a^{-\frac{t^k}{p}} dt.$$

By using the integral representation of generalized $p-k$ gamma function ${}_p^a\Gamma_k(x)$, we obtain the equation (3.2). ☐

Corollary 3.3. *Let m, n, x, y be positive numbers with the property that*

$$(m-x)(y-n) \geq (\leq) 0.$$

Then

$${}_p^aB_k(x,y)\, {}_p^aB_k(m,n) \geq (\leq) {}_p^aB_k(x,n)\, {}_p^aB_k(m,y). \tag{3.3}$$

Proof. Letting $i = 0$ in the inequality (3.2), we get

$${}^{a}_{p}\Gamma_k(x+n)\,{}^{a}_{p}\Gamma_k(y+m) \geq (\leq){}^{a}_{p}\Gamma_k(x+y)\,{}^{a}_{p}\Gamma_k(m+n).$$

Then we can write

$$\frac{{}^{a}_{p}\Gamma_k(x+n)\,{}^{a}_{p}\Gamma_k(y+m)}{{}^{a}_{p}\Gamma_k(x)\,{}^{a}_{p}\Gamma_k(n)\,{}^{a}_{p}\Gamma_k(y)\,{}^{a}_{p}\Gamma_k(m)} \geq (\leq)\frac{{}^{a}_{p}\Gamma_k(x+y)\,{}^{a}_{p}\Gamma_k(m+n)}{{}^{a}_{p}\Gamma_k(x)\,{}^{a}_{p}\Gamma_k(n)\,{}^{a}_{p}\Gamma_k(y)\,{}^{a}_{p}\Gamma_k(m)}.$$

Now by using the definition of generalized $p-k$ beta function ${}^{a}_{p}B_k(x,y)$, we get

$$\frac{1}{{}^{a}_{p}B_k(x,n)\,{}^{a}_{p}B_k(y,m)} \geq (\leq)\frac{1}{{}^{a}_{p}B_k(x,y)\,{}^{a}_{p}B_k(m,n)}$$

and the result follows. □

By using Theorem 3.2, we obtain the geometric means of ${}^{a}_{p}B_k(x,x)$ and ${}^{a}_{p}B_k(y,y)$ as follows:

Corollary 3.4. *For any $x, y > 0$, we have the inequality*

$$\sqrt{{}^{a}_{p}B_k(x,x){}^{a}_{p}B_k(y,y)} \leq {}^{a}_{p}B_k(x,y). \tag{3.4}$$

Proof. Letting $m = y$ and $n = x$ in Corollary 3.3 and using the symmetry property of generalized $p-k$ beta function ${}^{a}_{p}B_k(x,y)$, the inequality (3.4) follows. □

Remark 3.1. Taking $a = e$, $p = k = 1$ and $i = 0$ in Theorem 3.2, Corollary 3.3 and 3.4 leads us the inequalities (3.6), (3.5) and (3.7) in [6, Theorem 1 and Corollary 1] respectively.

Corollary 3.5. *Let $m, n > 0$ and i be a non-negative integer. Then we have*

$$\sqrt{{}^{a}_{p}\Gamma^{(2i)}(m){}^{a}_{p}\Gamma^{(2i)}(n)} \leq {}^{a}_{p}\Gamma^{(2i)}\left(\frac{m+n}{2}\right) \tag{3.5}$$

for $p, k > 0$ and $a \in (1, \infty)$.

Proof. Letting $m = y$ and $n = x$ in the Theorem 3.2 we have $(y-x)^2 \geq 0$. From the inequality (3.2), we get

$${}^{a}_{p}\Gamma^{(2i)}(2x){}^{a}_{p}\Gamma^{(2i)}(2y) \geq \left[{}^{a}_{p}\Gamma^{(2i)}(x+y)\right]^2. \tag{3.6}$$

Now the inequality (3.5) follows from (3.6) by taking $m = 2x$ and $n = 2y$. □

Theorem 3.6. *Let x, y, m be real number with $x, y > 0$, $x > m > -y$ and i be a non-negative integer. If*

$$m(x-y-m) \geq (\leq)0,$$

then the following inequality

$${}^{a}_{p}\Gamma^{(2i)}_k(x){}^{a}_{p}\Gamma^{(2i)}_k(y) \geq (\leq){}^{a}_{p}\Gamma^{(2i)}_k(x-m){}^{a}_{p}\Gamma^{(2i)}_k(y+m) \tag{3.7}$$

is valid.

Proof. Let us define $f, g, h : [0,\infty) \to [0,\infty)$ by $f(t) = t^{x-m-y}$, $g(t) = t^m$ and $h(t) = t^{y-1}\ln^{2i} t\, a^{-\frac{t^k}{p}}$ respectively. From the assumption, we get that the functions f and g are synchronous (asynchronous) on $[0,\infty)$. Hence by Chebychev's inequality for $I = [0,\infty)$ and the integral representation of generalized $p-k$ gamma function, we obtain

$$\int_0^\infty t^{y-1}\ln^{2i} t\, a^{-\frac{t^k}{p}}\, dt \int_0^\infty t^{x-1}\ln^{2i} t\, a^{-\frac{u^k}{p}}\, dt$$

$$\geq (\leq) \int_0^\infty t^{x-m-1}\ln^{2i} t\, a^{-\frac{t^k}{p}}\, dt \int_0^\infty t^{y+m-1}\ln^{2i} t\, a^{-\frac{t^k}{p}}\, dt$$

and the result follows. □

Corollary 3.7. *Let $x > 0$ and $m \in \mathbb{R}$ be such that $|m| < x$. Then for a non-negative integer i, we have*

$$\left[{}_p^a\Gamma_k^{(2i)}(x)\right]^2 \leq {}_p^a\Gamma_k^{(2i)}(x-m){}_p^a\Gamma_k^{(2i)}(x+m). \tag{3.8}$$

Proof. Letting $x = y$ in Theorem 3.6 yields that $m(x-x-m) = -m^2 < 0$ and the result follows. □

Corollary 3.8. *Let $x > 0$ and $m \in \mathbb{R}$ such that $|m| < x$. Then*

$${}_p^a\Gamma_k^2(x) \leq {}_p^a\Gamma_k(x-m){}_p^a\Gamma_k(x+m) \tag{3.9}$$

and

$${}_p^aB_k(x,x) \leq {}_p^aB_k(x-m,x+m) \tag{3.10}$$

Proof. For $i = 0$, the inequality (3.8) becomes (3.9). The inequality (3.10) follows from the Definition 2.1. □

Remark 3.2. By taking $p = k$, $a = e$, $i = 0$ in Theorem 3.6, Corollary 3.7 and Corollary 3.8, the inequalities become the results in [16, Theorem 2.6, Corollary 2.7, Corollary 2.8] respectively. Also by choosing $p = k = 1$, $a = e$, $i = 0$ in Theorem 3.6 and Corollary 3.8, we get [6, Theorem 2, Corollary 3] respectively.

Theorem 3.9. *Let $x, y, k > 0$ be such that $(x-k)(y-k) \geq (\leq)0$. Then for a non-negative integer i, we have*

$${}_p^a\Gamma_k^{(2i)}(2k){}_p^a\Gamma_k^{2i}(x+y) \geq (\leq){}_p^a\Gamma_k^{(2i)}(x+k){}_p^a\Gamma_k^{(2i)}(y+k). \tag{3.11}$$

Proof. Let us define $f, g, h : [0,\infty) \to [0,\infty)$ by $f(t) = t^{x-k}$, $g(t) = t^{y-k}$ and $h(t) = t^{2k-1}\ln^{2i} t a^{-\frac{t^k}{p}}$. Since $(x-k)(y-k) \geq (\leq)0$, the functions f and g are synchronous (asynchronous) on $[0,\infty)$. Hence by using Chebychev's inequality, we find

$$\int_0^\infty t^{2k-1}\ln^{2i} t\, a^{-\frac{t^k}{p}}\,dt \int_0^\infty t^{x+y-1}\ln^{2i} t\, a^{-\frac{t^k}{p}}\,dt \geq (\leq) \int_0^\infty t^{x+k-1}\ln^{2i} t\, a^{-\frac{t^k}{p}}\,dt \int_0^\infty t^{y+k-1}\ln^{2i} t\, a^{-\frac{t^k}{p}}\,dt$$

$${}_p^a\Gamma_k^{(2i)}(2k){}_p^a\Gamma_k^{2i}(x+y) \geq (\leq){}_p^a\Gamma_k^{(2i)}(x+k){}_p^a\Gamma_k^{(2i)}(y+k)$$

as desired. □

Corollary 3.10. *Let x, y, $k > 0$ such that $(x-k)(y-k) \geq (\leq)0$. Then the following inequalities*

$${}_p^a\Gamma_k(x+y) \geq (\leq)\frac{xy}{k}{}_p^a\Gamma_k(x){}_p^a\Gamma_k(y) \tag{3.12}$$

and

$${}_p^aB_k(x,y) \geq (\leq)\frac{k}{xy} \tag{3.13}$$

are valid.

Proof. By taking $i = 0$ in the inequality (3.11) and using the relation ${}_p^a\Gamma_k(x+k) = \dfrac{xp}{k\log a}{}_p^a\Gamma_k(x)$, given in [11], we get

$$\frac{p}{\log a}{}_p^a\Gamma_k(k){}_p^a\Gamma_k(x+y) \quad \geq (\leq) \quad \frac{xp}{k\log a}{}_p^a\Gamma_k(x)\frac{yp}{k\log a}{}_p^a\Gamma_k(y).$$

Since ${}_p^a\Gamma_k(k) = \dfrac{p}{k\log a}$, the inequality (3.12) follows and by the definition of generalized $p-k$ beta function ${}_p^aB_k(x,y)$ we get the inequality (3.13). □

Note that, a function f is said to be superadditive if

$$f(x+y) \geq f(x) + f(y)$$

and supermultiplicative if

$$f(xy) \geq f(x)f(y)$$

hold for all $x, y \in I$ such that $x+y \in I$ and $xy \in I$ respectively.

Corollary 3.11. *The function* $\ln {}_p^a\Gamma_k(x)$ *is superadditive for* $x \geq k \geq 1$.

Proof. If $x, y \geq k \geq 1$, then by using the inequality (3.12), we have

$$\ln {}_p^a\Gamma_k(x+y) \geq \ln x + \ln y - \ln k + \ln {}_p^a\Gamma_k(x) + \ln {}_p^a\Gamma_k(y) \geq \ln {}_p^a\Gamma_k(x) + \ln {}_p^a\Gamma_k(y)$$

which completes the proof. $\square$

4 Inequalities via the Hölder's One and Applications

Firstly, we recall the Hölder's inequality:

Lemma 4.1. *Let* $I \subset \mathbb{R}$ *be an interval and assume that* $f \in L_p(I)$, $g \in L_p(I)$, $p, q > 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then the following inequality due to Hölder holds:*

$$\left| \int_I f(x)g(x)dx \right| \leq \left(\int_I |f(x)|^p dx \right)^{1/p} \left(\int_I |g(x)|^q dx \right)^{1/q}. \tag{4.1}$$

For the proof see the book [18].

Now, by using the Hölder's inequality (4.1), we establish some inequalities of the functions ${}_p^a\Gamma_k$ and ${}_p^aB_k$.

Theorem 4.2. *Let* $x, y, k > 0$, $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$, i, j *even,* $i, j \in \mathbb{N}_0$ *and* $\alpha i + \beta j \in \mathbb{N}_0$. *Then generalized* $p-k$ *gamma function satisfies the inequality*

$${}_p^a\Gamma_k^{(\alpha i+\beta j)}(\alpha x + \beta y) \leq \left[{}_p^a\Gamma_k^{(i)}(x) \right]^\alpha \left[{}_p^a\Gamma_k^{(j)}(y) \right]^\beta. \tag{4.2}$$

Proof. By using integral representation of the generalized $p-k$ gamma function, we get

$${}_p^a\Gamma_k^{(\alpha i+\beta j)}(\alpha x + \beta y) = \int_0^\infty t^{\alpha x + \beta y - 1} \ln^{\alpha i + \beta j} t a^{-\frac{t^k}{p}} dt.$$

Then since $\alpha + \beta = 1$ and i, j are even, we have

$$\begin{aligned} {}_p^a\Gamma_k^{(\alpha i+\beta j)}(\alpha x + \beta y) &= \int_0^\infty t^{\alpha(x-1)} \ln^{\alpha i} t a^{-\frac{t^k}{p}\alpha} t^{\beta(y-1)} \ln^{\beta j} t a^{-\frac{t^k}{p}\beta} dt \\ &\leq \left[\int_0^\infty t^{x-1} \ln^i t a^{-\frac{t^k}{p}} dt \right]^\alpha \left[\int_0^\infty t^{y-1} \ln^j t a^{-\frac{t^k}{p}} dt \right]^\beta, \end{aligned}$$

by using the Hölder's inequality (4.1) and the result follows. $\square$

Now we will give the following well known definition in the literature, see for example [17].

Definition 4.1. Let $f : [a, b] \subset \mathbb{R} \to (0, \infty)$. Then f is called a log-convex function, if

$$f(\alpha x + (1 - \alpha)y) \leq [f(x)]^{\alpha}[f(y)]^{1-\alpha}$$

holds for any $x, y \in [a, b]$ and $\alpha \in [0, 1]$.

Corollary 4.3. *Let $x > 0$, $b \geq 0$, $p, k > 0$, $a \in (0, 1)$, $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$, i even $i \in \mathbb{N}_0$. Then the function ${}_p^a\Gamma_k^{(i)}(x)$ is log-convex.*

Proof. From Theorem 4.2 by letting $i = j$, we have

$${}_p^a\Gamma_k^{(i)}(\alpha x + \beta y) \leq \left[{}_p^a\Gamma_k^{(i)}(x)\right]^{\alpha} \left[{}_p^a\Gamma_k^{(i)}(y)\right]^{\beta}$$

i.e. the function ${}_p^a\Gamma_k^{(i)}$ is log-convex. □

Corollary 4.4. *The function ${}_p^aB_k$ is log-covex on $(0, \infty) \times (0, \infty)$ as a function of two variables for $p, k > 0$ and $a \in (1, \infty)$.*

Proof. Let $((x, y), (m, n)) \in (0, \infty) \times (0, \infty)$ and α, $\beta \geq 0$ such that $\alpha + \beta = 1$. Then we have

$$\begin{aligned} {}_p^aB_k(\alpha(x + y) + \beta(m, n)) &= {}_p^aB_k(\alpha x + \beta m, \alpha y + \beta n) \\ &= \frac{1}{k}\int_0^1 t^{\frac{\alpha x+\beta m}{k}-1}(1 - t)^{\frac{\alpha y+\beta n}{k}-1}dt. \end{aligned}$$

Then since $\alpha + \beta = 1$, we can write

$$\begin{aligned} {}_p^aB_k(\alpha(x + y) + \beta(m, n)) &= \frac{1}{k}\int_0^1 t^{\alpha\left(\frac{x}{k}-1\right)}(1 - t)^{\alpha\left(\frac{y}{k}-1\right)}t^{\beta\left(\frac{m}{k}-1\right)}(1 - t)^{\beta\left(\frac{n}{k}-1\right)}dt \\ &= \frac{1}{k}\int_0^1 \left[t^{\left(\frac{x}{k}-1\right)}(1 - t)^{\left(\frac{y}{k}-1\right)}\right]^{\alpha} \left[t^{\left(\frac{m}{k}-1\right)}(1 - t)^{\left(\frac{n}{k}-1\right)}\right]^{\beta} dt. \end{aligned}$$

Now by using the inequality (4.1), we have

$$\begin{aligned} {}_p^aB_k(\alpha(x + y) + \beta(m, n)) &\leq \frac{1}{k}[k{}_p^aB_k(x, y)]^{\alpha} [k{}_p^aB_k(m, n)]^{\beta} \\ &= k^{\alpha+\beta-1}[{}_p^aB_k(x, y)]^{\alpha} [{}_p^aB_k(m, n)]^{\beta}. \end{aligned}$$

By taking $\alpha = \lambda$ and $\beta = 1 - \lambda$, $\lambda \in (0, 1)$, we get the result. □

5 Conclusions

In this study, we establish some inequalities for the generalized $p - k$ gamma and beta functions by using Chebychev's and Hölder's inequalities and other algebraic tools. The established results are generalizations of some previous results.

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Competing Interests

Authors have declared that no competing interests exist.

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