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The Numerical Application of Dynamic Problems Involving Mass in Motion Governed by Higher Order Oscillatory Differential Equations

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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ABSTRACT

Real-world problems, particularly in the sciences and engineering, are often analyzed using differential equations to understand physical phenomena. Many situations involve rates of change of independent variables, represented by derivatives, which lead to differential equations. Solving higher-order ordinary differential equations typically involves reducing them to systems of first-order equations, but this approach has challenges. To overcome these and enhance numerical methods, a novel one-step block method with eight partitions was developed for the direct solution of higher-order initial value problems. This method will target issues in physics, biology, chemistry and economics. The new method was formulated using the linear block approach and numerical analysis was ensure essential and sufficient conditions. The new method addresses second-order problems like simple harmonic motion, third-order issues such as oscillatory differential equations,

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and fourth-order problems like thin film dynamics. The new method demonstrates faster convergence and improved accuracy compared to existing solutions for second, third, and fourth-order oscillatory differential equations.

Keywords: Harmonic motion; linear block approach; methodological technique; oscillatory differential equation; thin film dynamics.

1. INTRODUCTION

In science and engineering, the majority of problems modeled using ordinary differential equations lack analytical solutions, necessitating the consideration of approximate solutions. These approximations are derived through numerical methods, resulting in the discretization of solutions. Discretization involves representing the solution as function values at grid points, which are connected through interpolation of the function as discussed in Ref. [1,2].

The subsequent oscillatory differential equations that lead to advanced order of differential equation of the form

$$u^{n} = f\left(v, u, u', u'', \cdots, u^{n}\right), \quad u^{(m-1)}(v_0) = \mu_{m-1}, n = 1, 2, \cdots, n \qquad (1)$$

are consider in this study. It is of great significant to researchers for mathematical models as differential equation.

Many physical problems are still unexplored and not fully addressed by researchers. Although certain issues in science, social science, and technology have been investigated, numerous others remain uncharted. Oscillatory phenomena frequently play a crucial role in these fields, and differential equations are one of the primary tools used to model such oscillations as discussed in [1,2].

There are two existing procedures for simulation of (1). The first procedure is to reduce (1) to the corresponding first-order system and then solve it using first-order ordinary differential equations as discussed in Ref. [2,3]. The next procedure is using the direct method as recommended in Ref. [3-7]. On the other hand, the process of reducing the oscillatory differential equation (1) to a first-order system leads to some setbacks, such as computational burden which affects the performance of the method and time constraints, as discussed in Ref. [8-11].

Therefore, efforts have been made to develop some schemes that solve (1) directly using different methods. Among others are Ref. [12-15] have developed schemes that solve second-order oscillatory problems of the form

$$u'' = f(v, u, u'), \; u(v_0) = \mu_0, \; u'(v_1) = \mu_1 \qquad (2)$$

Similarly, approaches have been proposed by Ref. [16-19] for solving

$$u''' = f\left(v, u, u, u''\right), u(v_0) = \mu_0, \; u'(v_1) = \mu_1, u''(v_2) = \mu_2 \qquad (3)$$

Lastly, some researchers [11, 20-23] directly employ their methods on

$$u'''' = f\left(v, u, u, u'', u'''\right), u(v_0) = \mu_0, \; u'(v_1) = \mu_1, u''(v_2) = \mu_2, u'''(v_3) = \mu_3 \qquad (4)$$

From this time, the new method will cater the setbacks by solving (2), (3) and (4) direct. Ref. [24-27] proposed a methods for direct solutions of (2), (3) and (4), the accuracy of the method is computational reliable.

2. CONSTRUCTION OF LINEAR BLOCK APPROACH

2.1 The $k-step$ Generalized Algorithm

The linear block approach were applied on derivation of new method for direct solution of higher order oscillatory differential equation (1) where $Y_{n+k} = (y_{n+a}, y_{n+b}, \cdots, y_{n+k})$ and $Y_{n+k}^{(j)} = \left(y_{n+a}^{(j)}, y_{n+b}^{(j)}, \cdots, y_{n+k}^{(j)}\right)$. In order to obtain the unknown values, the generalized algorithm

$$y_{n+\xi} = \sum_{j=0}^{3} \frac{(\xi h)^j}{j!} y_n^{(j)} + \sum_{j=0}^{k} \left(\psi_{i\xi} f_{n+j}\right), \quad \zeta = a, b, \cdots, k \qquad (5)$$

its higher derivatives

$$y_{n+\xi}^{\varsigma} = \sum_{j=0}^{4-(\varsigma+1)} \frac{(\xi h)^j}{j!} y_n^{(j+\varsigma)} +$$

$$\varsigma = 1_{(\xi=a,b,\cdots,k)},\ \varsigma = 2_{(\xi=a,b,\cdots,k)},\ \varsigma = 3_{(\xi=a,b,\cdots,k)} \quad (6)$$

is consider, with $\psi_{\xi j} = U^{-1}G$ and $\Omega_{\xi j \varsigma} = U^{-1}D$ where

$$U = \begin{pmatrix} 1 & 1 & 1 & \cdots & k \\ 0 & \frac{(ah)^1}{1!} & \frac{(bh)^1}{1!} & \cdots & \frac{(kh)^1}{1!} \\ 0 & \frac{(ah)^2}{2!} & \frac{(bh)^2}{2!} & \cdots & \frac{(kh)^2}{2!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{(ah)^m}{m!} & \frac{(bh)^m}{m!} & \cdots & \frac{(kh)^m}{m!} \end{pmatrix},$$

$$G = \begin{pmatrix} \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \vdots \\ \frac{(\xi h)^{(4+m)}}{(4+m)!} \end{pmatrix},\ D = \begin{pmatrix} \frac{(\xi h)^{4-\varsigma}}{(4-\varsigma)!} \\ \frac{(\xi h)^{(5-\varsigma)+a}}{((5-\varsigma)+a)!} \\ \frac{(\xi h)^{(6-\varsigma)+b}}{((6-\varsigma)+b)!} \\ \vdots \\ \frac{(\xi h)^{(m-\varsigma)+k}}{((m-\varsigma)+k)!} \end{pmatrix}$$

So, to derive the new methods, the subsequent Corollary were proved.

Corollary 1

The $k-step$ general linear multistep method associated with a linear block approach (5) and (5) adopts only a block method. The corollary is generalized to develop the higher order scheme from block algorithm.

This can be verified with the help of the equation (5) and (6) as a block at the points $\left(0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, 1\right)$.

Proof

Now simplifying (5) and (6) using the partitioned points, we have

$$U = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{h}{8} & \frac{h}{4} & \frac{3h}{8} & \frac{h}{2} & \frac{5h}{8} & \frac{3h}{4} & \frac{7h}{8} & h \\ 0 & \frac{\left(\frac{h}{8}\right)^2}{2!} & \frac{\left(\frac{2h}{8}\right)^2}{2!} & \frac{\left(\frac{3h}{8}\right)^2}{2!} & \frac{\left(\frac{4h}{8}\right)^2}{2!} & \frac{\left(\frac{5h}{8}\right)^2}{2!} & \frac{\left(\frac{6h}{8}\right)^2}{2!} & \frac{\left(\frac{7h}{8}\right)^2}{2!} & \frac{(h)^2}{2!} \\ 0 & \frac{\left(\frac{h}{8}\right)^3}{3!} & \frac{\left(\frac{2h}{8}\right)^3}{3!} & \frac{\left(\frac{3h}{8}\right)^3}{3!} & \frac{\left(\frac{4h}{8}\right)^3}{3!} & \frac{\left(\frac{5h}{8}\right)^3}{3!} & \frac{\left(\frac{6h}{8}\right)^3}{3!} & \frac{\left(\frac{7h}{8}\right)^3}{3!} & \frac{(h)^3}{3!} \\ 0 & \frac{\left(\frac{h}{8}\right)^4}{4!} & \frac{\left(\frac{2h}{8}\right)^4}{4!} & \frac{\left(\frac{3h}{8}\right)^4}{4!} & \frac{\left(\frac{4h}{8}\right)^4}{4!} & \frac{\left(\frac{5h}{8}\right)^4}{4!} & \frac{\left(\frac{6h}{8}\right)^4}{4!} & \frac{\left(\frac{7h}{8}\right)^4}{4!} & \frac{(h)^4}{4!} \\ 0 & \frac{\left(\frac{h}{8}\right)^5}{5!} & \frac{\left(\frac{2h}{8}\right)^5}{5!} & \frac{\left(\frac{3h}{8}\right)^5}{5!} & \frac{\left(\frac{4h}{8}\right)^5}{5!} & \frac{\left(\frac{5h}{8}\right)^5}{5!} & \frac{\left(\frac{6h}{8}\right)^5}{5!} & \frac{\left(\frac{7h}{8}\right)^5}{5!} & \frac{(h)^5}{5!} \\ 0 & \frac{\left(\frac{h}{8}\right)^6}{6!} & \frac{\left(\frac{2h}{8}\right)^6}{6!} & \frac{\left(\frac{3h}{8}\right)^6}{6!} & \frac{\left(\frac{4h}{8}\right)^6}{6!} & \frac{\left(\frac{5h}{8}\right)^6}{6!} & \frac{\left(\frac{6h}{8}\right)^6}{6!} & \frac{\left(\frac{7h}{8}\right)^6}{6!} & \frac{(h)^6}{6!} \\ 0 & \frac{\left(\frac{h}{8}\right)^7}{7!} & \frac{\left(\frac{2h}{8}\right)^7}{7!} & \frac{\left(\frac{3h}{8}\right)^7}{7!} & \frac{\left(\frac{4h}{8}\right)^7}{7!} & \frac{\left(\frac{5h}{8}\right)^7}{7!} & \frac{\left(\frac{6h}{8}\right)^7}{7!} & \frac{\left(\frac{7h}{8}\right)^7}{7!} & \frac{(h)^7}{7!} \\ 0 & \frac{\left(\frac{h}{8}\right)^8}{8!} & \frac{\left(\frac{2h}{8}\right)^8}{8!} & \frac{\left(\frac{3h}{8}\right)^8}{8!} & \frac{\left(\frac{4h}{8}\right)^8}{8!} & \frac{\left(\frac{5h}{8}\right)^8}{8!} & \frac{\left(\frac{6h}{8}\right)^8}{8!} & \frac{\left(\frac{7h}{8}\right)^8}{8!} & \frac{(h)^8}{8!} \end{pmatrix},$$

$$G = \begin{pmatrix} \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \frac{(\xi h)^7}{7!} \\ \frac{(\xi h)^8}{8!} \\ \frac{(\xi h)^9}{9!} \\ \frac{(\xi h)^{10}}{10!} \\ \frac{(\xi h)^{11}}{11!} \\ \frac{(\xi h)^{12}}{12!} \end{pmatrix},\ D = \begin{pmatrix} \frac{(\xi h)^{4-\tau}}{(4-\tau)!} \\ \frac{(\xi h)^{5-\tau}}{(5-\tau)!} \\ \frac{(\xi h)^{6-\tau}}{(6-\tau)!} \\ \frac{(\xi h)^{7-\tau}}{(7-\tau)!} \\ \frac{(\xi h)^{8-\tau}}{(8-\tau)!} \\ \frac{(\xi h)^{9-\tau}}{(9-\tau)!} \\ \frac{(\xi h)^{10-\tau}}{(10-\tau)!} \\ \frac{(\xi h)^{11-\tau}}{(11-\tau)!} \\ \frac{(\xi h)^{12-\tau}}{(12-\tau)!} \end{pmatrix}$$

Solving equations (5) and (6) one by one to obtain the coefficients of the polynomial $y_{n\xi},\ \xi = 0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, 1$

Substituting $\xi = \xi_n + xh$, the polynomial takes the form

$$y(\xi_n + xh) = \alpha_{\frac{1}{8}} y_{n+\frac{1}{8}} + \alpha_{\frac{3}{8}} y_{n+\frac{3}{8}} + \alpha_{\frac{5}{8}} y_{n+\frac{5}{8}} + \alpha_{\frac{7}{8}} y_{n+\frac{7}{8}} + h^4 \begin{pmatrix} \beta_0 f_n + \beta_{\frac{1}{8}} f_{n+\frac{1}{8}} + \beta_{\frac{2}{8}} f_{n+\frac{2}{8}} + \beta_{\frac{3}{8}} f_{n+\frac{3}{8}} + \beta_{\frac{4}{8}} f_{n+\frac{4}{8}} \\ + \beta_{\frac{5}{8}} f_{n+\frac{5}{8}} + \beta_{\frac{6}{8}} f_{n+\frac{6}{8}} + \beta_{\frac{7}{8}} f_{n+\frac{7}{8}} + \beta_1 f_{n+1} \end{pmatrix} \quad (7)$$

Where

$$\alpha_{\frac{1}{8}}=\frac{35}{16}-\frac{71}{6}\xi+20\xi^2-\frac{32}{16}\xi^3,\ \alpha_{\frac{3}{8}}=-\frac{35}{16}+\frac{47}{2}\xi+52\xi^2+32\xi^3,\ \alpha_{\frac{5}{8}}=\frac{21}{16}-\frac{31}{2}\xi+44\xi^2-32\xi^3,\ \alpha_{\frac{7}{8}}=-\frac{5}{16}+\frac{23}{6}\xi-12\xi^2+\frac{32}{3}\xi^3$$

$$\beta_0=-\frac{37}{3397386240}-\frac{969961}{6131220480\ 0}\xi+\frac{14612201}{30\ 656102400}\xi^2-\frac{1041329}{1041329}\xi^3+\frac{1}{24}\xi^4-\frac{761}{4200}\xi^5+\frac{29531}{56700}\xi^6-\frac{178}{175}\xi^7+\frac{2138}{1575}\xi^8$$
$$-\frac{128}{105}\xi^9+\frac{3328}{4725}\xi^{10}-\frac{4096}{17325}\xi^{11}+\frac{16384}{467775}\xi^{12}$$

$$\beta_{\frac{1}{8}}=\frac{1675}{42467328}-\frac{11621213}{12\ 262440960}\xi+\frac{9416159}{1094860800}\xi^2-\frac{115331}{3483648}\xi^3+\frac{8}{15}\xi^5-\frac{3848}{1575}\xi^6+\frac{5584}{945}\xi^7-\frac{6016}{675}\xi^8+\frac{14720}{1701}\xi^9-$$
$$\frac{74752}{14175}\xi^{10}+\frac{8192}{4455}\xi^{11}-\frac{131072}{467775}\xi^{12}$$

$$\beta_{\frac{2}{8}}=\frac{41041}{169869312}-\frac{3031321}{875888640}\xi+\frac{100608941}{7664025600}\xi^2-\frac{3151}{9676800}\xi^3-\frac{14}{15}\xi^5+\frac{138}{25}\xi^6-\frac{73412}{4725}\xi^7+\frac{122312}{4725}\xi^8-\frac{45824}{1701}\xi^9+$$
$$\frac{244736}{14175}\xi^{10}-\frac{139264}{22275}\xi^{11}+\frac{65536}{66825}\xi^{12}$$

$$\beta_{\frac{3}{8}}=\frac{16081}{42467328}-\frac{5459059}{1114767360}\xi+\frac{147496847}{7664025600}\xi^2-\frac{537337}{12441600}\xi^3+\frac{56}{45}\xi^5-\frac{16024}{2025}\xi^6+\frac{12752}{525}\xi^7-\frac{68608}{1575}\xi^8+\frac{9088}{189}\xi^9-$$
$$\frac{152576}{4725}\xi^{10}+\frac{8192}{675}\xi^{11}+\frac{131072}{66825}\xi^{12}$$

$$\beta_{\frac{4}{8}}=\frac{100541}{339738624}-\frac{1574633}{437944320}\xi+\frac{5615635}{613122048}\xi^2+\frac{17010}{227}\xi^3-\frac{7}{6}\xi^5+\frac{691}{90}\xi^6-\frac{23312}{945}\xi^7+\frac{43972}{945}\xi^8-\frac{91648}{1701}\xi^9+$$
$$\frac{107008}{2835}\xi^{10}-\frac{65536}{4455}\xi^{11}+\frac{32768}{13365}\xi^{12}$$

$$\beta_{\frac{5}{8}}=\frac{21017}{212336640}-\frac{76865659}{61\ 312204800}\xi+\frac{5615635}{7664025600}\xi^2-\frac{179177}{9676800}\xi^3+\frac{56}{75}\xi^5-\frac{376}{75}\xi^6+\frac{78256}{4725}\xi^7-\frac{152704}{4725}\xi^8+\frac{330368}{8505}\xi^9$$
$$-\frac{400384}{14175}\xi^{10}+\frac{253952}{22275}\xi^{11}-\frac{131072}{66825}\xi^{12}$$

$$\beta_{\frac{6}{8}}=\frac{2353}{169869312}-\frac{937273}{6131220480}\xi-\frac{954763}{7664025600}\xi^2-\frac{100699}{17418240}\xi^3+\frac{14}{45}\xi^5-\frac{4286}{2025}\xi^6+\frac{748}{105}\xi^7-\frac{22424}{1575}\xi^8+\frac{3328}{189}\xi^9$$
$$-\frac{62464}{4725}\xi^{10}+\frac{8192}{1485}\xi^{11}-\frac{65536}{66825}\xi^{12}$$

$$\beta_{\frac{7}{8}}=-\frac{17}{42467328}+\frac{1577}{2452488192}\xi+\frac{1114049}{7664025600}\xi^2-\frac{131381}{87091200}\xi^3+\frac{8}{105}\xi^5-\frac{824}{1575}\xi^6+\frac{8432}{4725}\xi^7-\frac{17152}{4725}\xi^8+\frac{7808}{1701}\xi^9$$
$$-\frac{7168}{2025}\xi^{10}+\frac{237568}{155925}\xi^{11}-\frac{131072}{131072}\xi^{12}$$

$$\beta_1=\frac{31}{679477248}-\frac{211}{2452488192}\xi-\frac{484903}{30\ 656102400}\xi^2+\frac{353}{2150400}\xi^3-\frac{1}{120}\xi^5+\frac{121}{2100}\xi^6-\frac{134}{675}\xi^7+\frac{1934}{4725}\xi^8-\frac{128}{243}\xi^9$$
$$+\frac{5888}{14175}\xi^{10}+\frac{4096}{22275}\xi^{11}-\frac{16348}{467775}\xi^{12}$$

The block algorithm (5) is expanded to yield

$$\left.\begin{aligned}
y_{n+\frac{1}{8}} &= y_n + \frac{1}{8}hy'_n + \frac{\left(\frac{1}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{1}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\psi_{011}f_n + \psi_{012}f_{n+\frac{1}{8}} + \psi_{013}f_{n+\frac{1}{4}} + \psi_{014}f_{n+\frac{3}{8}} + \psi_{015}f_{n+\frac{1}{2}}\\ +\psi_{016}f_{n+\frac{5}{8}} + \psi_{017}f_{n+\frac{3}{4}} + \psi_{018}f_{n+\frac{7}{8}} + \psi_{019}f_{n+1}\end{pmatrix}\\
y_{n+\frac{2}{8}} &= y_n + \frac{2}{8}hy'_n + \frac{\left(\frac{2}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{2}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\psi_{021}f_n + \psi_{022}f_{n+\frac{1}{8}} + \psi_{023}f_{n+\frac{1}{4}} + \psi_{024}f_{n+\frac{3}{8}} + \psi_{025}f_{n+\frac{1}{2}}\\ +\psi_{026}f_{n+\frac{5}{8}} + \psi_{027}f_{n+\frac{3}{4}} + \psi_{028}f_{n+\frac{7}{8}} + \psi_{029}f_{n+1}\end{pmatrix}\\
y_{n+\frac{3}{8}} &= y_n + \frac{3}{8}hy'_n + \frac{\left(\frac{3}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{3}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\psi_{031}f_n + \psi_{032}f_{n+\frac{1}{8}} + \psi_{033}f_{n+\frac{1}{4}} + \psi_{034}f_{n+\frac{3}{8}} + \psi_{035}f_{n+\frac{1}{2}}\\ +\psi_{036}f_{n+\frac{5}{8}} + \psi_{037}f_{n+\frac{3}{4}} + \psi_{038}f_{n+\frac{7}{8}} + \psi_{039}f_{n+1}\end{pmatrix}\\
y_{n+\frac{4}{8}} &= y_n + \frac{4}{8}hy'_n + \frac{\left(\frac{4}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{4}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\psi_{041}f_n + \psi_{042}f_{n+\frac{1}{8}} + \psi_{043}f_{n+\frac{1}{4}} + \psi_{044}f_{n+\frac{3}{8}} + \psi_{045}f_{n+\frac{1}{2}}\\ +\psi_{046}f_{n+\frac{5}{8}} + \psi_{047}f_{n+\frac{3}{4}} + \psi_{048}f_{n+\frac{7}{8}} + \psi_{049}f_{n+1}\end{pmatrix}\\
y_{n+\frac{5}{8}} &= y_n + \frac{5}{8}hy'_n + \frac{\left(\frac{5}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{5}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\psi_{051}f_n + \psi_{052}f_{n+\frac{1}{8}} + \psi_{053}f_{n+\frac{1}{4}} + \psi_{054}f_{n+\frac{3}{8}} + \psi_{055}f_{n+\frac{1}{2}}\\ +\psi_{056}f_{n+\frac{5}{8}} + \psi_{057}f_{n+\frac{3}{4}} + \psi_{058}f_{n+\frac{7}{8}} + \psi_{059}f_{n+1}\end{pmatrix}\\
y_{n+\frac{6}{8}} &= y_n + \frac{6}{8}hy'_n + \frac{\left(\frac{6}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{6}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\psi_{061}f_n + \psi_{062}f_{n+\frac{1}{8}} + \psi_{063}f_{n+\frac{1}{4}} + \psi_{064}f_{n+\frac{3}{8}} + \psi_{065}f_{n+\frac{1}{2}}\\ +\psi_{066}f_{n+\frac{5}{8}} + \psi_{067}f_{n+\frac{3}{4}} + \psi_{068}f_{n+\frac{7}{8}} + \psi_{069}f_{n+1}\end{pmatrix}\\
y_{n+\frac{7}{8}} &= y_n + \frac{7}{8}hy'_n + \frac{\left(\frac{7}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{7}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\psi_{071}f_n + \psi_{072}f_{n+\frac{1}{8}} + \psi_{073}f_{n+\frac{1}{4}} + \psi_{074}f_{n+\frac{3}{8}} + \psi_{075}f_{n+\frac{1}{2}}\\ +\psi_{076}f_{n+\frac{5}{8}} + \psi_{077}f_{n+\frac{3}{4}} + \psi_{078}f_{n+\frac{7}{8}} + \psi_{079}f_{n+1}\end{pmatrix}\\
y_{n+1} &= y_n + hy'_n + \frac{(h)^2}{2!}y''_n + \frac{(h)^3}{3!}y'''_n + h^4\begin{pmatrix}\psi_{081}f_n + \psi_{082}f_{n+\frac{1}{8}} + \psi_{083}f_{n+\frac{1}{4}} + \psi_{084}f_{n+\frac{3}{8}} + \psi_{085}f_{n+\frac{1}{2}}\\ +\psi_{086}f_{n+\frac{5}{8}} + \psi_{087}f_{n+\frac{3}{4}} + \psi_{088}f_{n+\frac{7}{8}} + \psi_{089}f_{n+1}\end{pmatrix}
\end{aligned}\right\} \quad (8)$$

Likewise**,** the linear block algorithm (6) is expanded to yield the higher derivatives as

$$\left.\begin{aligned}
y'_{n+\frac{1}{8}} &= y'_n + \frac{1}{8}hy''_n + \frac{\left(\frac{1}{8}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{111}f_n + \Omega_{112}f_{n+\frac{1}{8}} + \Omega_{113}f_{n+\frac{1}{4}} + \Omega_{114}f_{n+\frac{3}{8}} + \Omega_{115}f_{n+\frac{1}{2}} + \Omega_{116}f_{n+\frac{5}{8}} + \Omega_{117}f_{n+\frac{3}{4}} + \Omega_{118}f_{n+\frac{7}{8}} + \Omega_{119}f_{n+1}\right)\\
y'_{n+\frac{1}{4}} &= y'_n + \frac{1}{4}hy''_n + \frac{\left(\frac{1}{4}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{121}f_n + \Omega_{122}f_{n+\frac{1}{8}} + \Omega_{123}f_{n+\frac{1}{4}} + \Omega_{124}f_{n+\frac{3}{8}} + \Omega_{125}f_{n+\frac{1}{2}} + \Omega_{126}f_{n+\frac{5}{8}} + \Omega_{127}f_{n+\frac{3}{4}} + \Omega_{128}f_{n+\frac{7}{8}} + \Omega_{129}f_{n+1}\right)\\
y'_{n+\frac{3}{8}} &= y'_n + \frac{3}{8}hy''_n + \frac{\left(\frac{3}{8}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{131}f_n + \Omega_{132}f_{n+\frac{1}{8}} + \Omega_{133}f_{n+\frac{1}{4}} + \Omega_{134}f_{n+\frac{3}{8}} + \Omega_{135}f_{n+\frac{1}{2}} + \Omega_{136}f_{n+\frac{5}{8}} + \Omega_{137}f_{n+\frac{3}{4}} + \Omega_{138}f_{n+\frac{7}{8}} + \Omega_{139}f_{n+1}\right)\\
y'_{n+\frac{1}{2}} &= y'_n + \frac{1}{2}hy''_n + \frac{\left(\frac{1}{2}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{141}f_n + \Omega_{142}f_{n+\frac{1}{8}} + \Omega_{143}f_{n+\frac{1}{4}} + \Omega_{144}f_{n+\frac{3}{8}} + \Omega_{145}f_{n+\frac{1}{2}} + \Omega_{146}f_{n+\frac{5}{8}} + \Omega_{147}f_{n+\frac{3}{4}} + \Omega_{148}f_{n+\frac{7}{8}} + \Omega_{149}f_{n+1}\right)\\
y'_{n+\frac{5}{8}} &= y'_n + \frac{5}{8}hy''_n + \frac{\left(\frac{5}{8}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{151}f_n + \Omega_{152}f_{n+\frac{1}{8}} + \Omega_{153}f_{n+\frac{1}{4}} + \Omega_{154}f_{n+\frac{3}{8}} + \Omega_{155}f_{n+\frac{1}{2}} + \Omega_{156}f_{n+\frac{5}{8}} + \Omega_{157}f_{n+\frac{3}{4}} + \Omega_{158}f_{n+\frac{7}{8}} + \Omega_{159}f_{n+1}\right)\\
y'_{n+\frac{3}{4}} &= y'_n + \frac{3}{4}hy''_n + \frac{\left(\frac{3}{4}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{161}f_n + \Omega_{162}f_{n+\frac{1}{8}} + \Omega_{163}f_{n+\frac{1}{4}} + \Omega_{164}f_{n+\frac{3}{8}} + \Omega_{165}f_{n+\frac{1}{2}} + \Omega_{166}f_{n+\frac{5}{8}} + \Omega_{167}f_{n+\frac{3}{4}} + \Omega_{168}f_{n+\frac{7}{8}} + \Omega_{119}f_{n+1}\right)\\
y'_{n+\frac{7}{8}} &= y'_n + \frac{7}{8}hy''_n + \frac{\left(\frac{7}{8}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{171}f_n + \Omega_{172}f_{n+\frac{1}{8}} + \Omega_{173}f_{n+\frac{1}{4}} + \Omega_{174}f_{n+\frac{3}{8}} + \Omega_{175}f_{n+\frac{1}{2}} + \Omega_{176}f_{n+\frac{5}{8}} + \Omega_{177}f_{n+\frac{3}{4}} + \Omega_{178}f_{n+\frac{7}{8}} + \Omega_{179}f_{n+1}\right)\\
y'_{n+1} &= y'_n + hy''_n + \frac{(h)^2}{2!}y'''_n + h^3\left(\Omega_{181}f_n + \Omega_{182}f_{n+\frac{1}{8}} + \Omega_{183}f_{n+\frac{1}{4}} + \Omega_{184}f_{n+\frac{3}{8}} + \Omega_{185}f_{n+\frac{1}{2}} + \Omega_{186}f_{n+\frac{5}{8}} + \Omega_{187}f_{n+\frac{3}{4}} + \Omega_{188}f_{n+\frac{7}{8}} + \Omega_{189}f_{n+1}\right)
\end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned}
y''_{n+\frac{1}{8}} &= y''_n + \frac{1}{8}hy'''_n + h^2\left(\Omega_{211}f_n + \Omega_{212}f_{n+\frac{1}{8}} + \Omega_{213}f_{n+\frac{1}{4}} + \Omega_{214}f_{n+\frac{3}{8}} + \Omega_{215}f_{n+\frac{1}{2}} + \Omega_{216}f_{n+\frac{5}{8}} + \Omega_{217}f_{n+\frac{3}{4}} + \Omega_{218}f_{n+\frac{7}{8}} + \Omega_{219}f_{n+1}\right)\\
y''_{n+\frac{1}{4}} &= y''_n + \frac{1}{4}hy'''_n + h^2\left(\Omega_{221}f_n + \Omega_{222}f_{n+\frac{1}{8}} + \Omega_{223}f_{n+\frac{1}{4}} + \Omega_{224}f_{n+\frac{3}{8}} + \Omega_{225}f_{n+\frac{1}{2}} + \Omega_{226}f_{n+\frac{5}{8}} + \Omega_{227}f_{n+\frac{3}{4}} + \Omega_{228}f_{n+\frac{7}{8}} + \Omega_{229}f_{n+1}\right)\\
y''_{n+\frac{3}{8}} &= y''_n + \frac{3}{8}hy'''_n + h^2\left(\Omega_{231}f_n + \Omega_{232}f_{n+\frac{1}{8}} + \Omega_{233}f_{n+\frac{1}{4}} + \Omega_{234}f_{n+\frac{3}{8}} + \Omega_{235}f_{n+\frac{1}{2}} + \Omega_{236}f_{n+\frac{5}{8}} + \Omega_{237}f_{n+\frac{3}{4}} + \Omega_{238}f_{n+\frac{7}{8}} + \Omega_{239}f_{n+1}\right)\\
y''_{n+\frac{1}{2}} &= y''_n + \frac{1}{2}hy'''_n + h^2\left(\Omega_{241}f_n + \Omega_{242}f_{n+\frac{1}{8}} + \Omega_{243}f_{n+\frac{1}{4}} + \Omega_{244}f_{n+\frac{3}{8}} + \Omega_{245}f_{n+\frac{1}{2}} + \Omega_{246}f_{n+\frac{5}{8}} + \Omega_{247}f_{n+\frac{3}{4}} + \Omega_{248}f_{n+\frac{7}{8}} + \Omega_{249}f_{n+1}\right)\\
y''_{n+\frac{5}{8}} &= y''_n + \frac{5}{8}hy'''_n + h^2\left(\Omega_{241}f_n + \Omega_{242}f_{n+\frac{1}{8}} + \Omega_{243}f_{n+\frac{1}{4}} + \Omega_{244}f_{n+\frac{3}{8}} + \Omega_{245}f_{n+\frac{1}{2}} + \Omega_{246}f_{n+\frac{5}{8}} + \Omega_{247}f_{n+\frac{3}{4}} + \Omega_{248}f_{n+\frac{7}{8}} + \Omega_{249}f_{n+1}\right)\\
y''_{n+\frac{3}{4}} &= y''_n + \frac{3}{4}hy'''_n + h^2\left(\Omega_{261}f_n + \Omega_{262}f_{n+\frac{1}{8}} + \Omega_{263}f_{n+\frac{1}{4}} + \Omega_{264}f_{n+\frac{3}{8}} + \Omega_{265}f_{n+\frac{1}{2}} + \Omega_{266}f_{n+\frac{5}{8}} + \Omega_{267}f_{n+\frac{3}{4}} + \Omega_{268}f_{n+\frac{7}{8}} + \Omega_{269}f_{n+1}\right)\\
y''_{n+\frac{7}{8}} &= y''_n + \frac{7}{8}hy'''_n + h^2\left(\Omega_{271}f_n + \Omega_{272}f_{n+\frac{1}{8}} + \Omega_{273}f_{n+\frac{1}{4}} + \Omega_{274}f_{n+\frac{3}{8}} + \Omega_{275}f_{n+\frac{1}{2}} + \Omega_{276}f_{n+\frac{5}{8}} + \Omega_{277}f_{n+\frac{3}{4}} + \Omega_{278}f_{n+\frac{7}{8}} + \Omega_{279}f_{n+1}\right)\\
y''_{n+1} &= y''_n + hy'''_n + h^2\left(\Omega_{281}f_n + \Omega_{282}f_{n+\frac{1}{8}} + \Omega_{283}f_{n+\frac{1}{4}} + \Omega_{284}f_{n+\frac{3}{8}} + \Omega_{285}f_{n+\frac{1}{2}} + \Omega_{286}f_{n+\frac{5}{8}} + \Omega_{287}f_{n+\frac{3}{4}} + \Omega_{288}f_{n+\frac{7}{8}} + \Omega_{289}f_{n+1}\right)
\end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned}
y'''_{n+\frac{1}{8}} &= y'''_n + h\left(\Omega_{311}f_n + \Omega_{312}f_{n+\frac{1}{8}} + \Omega_{313}f_{n+\frac{1}{4}} + \Omega_{314}f_{n+\frac{3}{8}} + \Omega_{315}f_{n+\frac{1}{2}} + \Omega_{316}f_{n+\frac{5}{8}} + \Omega_{317}f_{n+\frac{3}{4}} + \Omega_{318}f_{n+\frac{7}{8}} + \Omega_{319}f_{n+1}\right)\\
y'''_{n+\frac{1}{4}} &= y'''_n + h\left(\Omega_{321}f_n + \Omega_{312}f_{n+\frac{1}{8}} + \Omega_{323}f_{n+\frac{1}{4}} + \Omega_{324}f_{n+\frac{3}{8}} + \Omega_{325}f_{n+\frac{1}{2}} + \Omega_{326}f_{n+\frac{5}{8}} + \Omega_{327}f_{n+\frac{3}{4}} + \Omega_{328}f_{n+\frac{7}{8}} + \Omega_{329}f_{n+1}\right)\\
y'''_{n+\frac{3}{8}} &= y'''_n + h\left(\Omega_{331}f_n + \Omega_{332}f_{n+\frac{1}{8}} + \Omega_{333}f_{n+\frac{1}{4}} + \Omega_{334}f_{n+\frac{3}{8}} + \Omega_{335}f_{n+\frac{1}{2}} + \Omega_{336}f_{n+\frac{5}{8}} + \Omega_{337}f_{n+\frac{3}{4}} + \Omega_{338}f_{n+\frac{7}{8}} + \Omega_{339}f_{n+1}\right)\\
y'''_{n+\frac{1}{2}} &= y'''_n + h\left(\Omega_{341}f_n + \Omega_{342}f_{n+\frac{1}{8}} + \Omega_{343}f_{n+\frac{1}{4}} + \Omega_{344}f_{n+\frac{3}{8}} + \Omega_{345}f_{n+\frac{1}{2}} + \Omega_{346}f_{n+\frac{5}{8}} + \Omega_{347}f_{n+\frac{3}{4}} + \Omega_{348}f_{n+\frac{7}{8}} + \Omega_{349}f_{n+1}\right)\\
y'''_{n+\frac{5}{8}} &= y'''_n + h\left(\Omega_{351}f_n + \Omega_{352}f_{n+\frac{1}{8}} + \Omega_{353}f_{n+\frac{1}{4}} + \Omega_{354}f_{n+\frac{3}{8}} + \Omega_{355}f_{n+\frac{1}{2}} + \Omega_{356}f_{n+\frac{5}{8}} + \Omega_{357}f_{n+\frac{3}{4}} + \Omega_{358}f_{n+\frac{7}{8}} + \Omega_{359}f_{n+1}\right)\\
y'''_{n+\frac{3}{4}} &= y'''_n + h\left(\Omega_{361}f_n + \Omega_{362}f_{n+\frac{1}{8}} + \Omega_{363}f_{n+\frac{1}{4}} + \Omega_{364}f_{n+\frac{3}{8}} + \Omega_{365}f_{n+\frac{1}{2}} + \Omega_{366}f_{n+\frac{5}{8}} + \Omega_{367}f_{n+\frac{3}{4}} + \Omega_{368}f_{n+\frac{7}{8}} + \Omega_{369}f_{n+1}\right)\\
y'''_{n+\frac{7}{8}} &= y'''_n + h\left(\Omega_{371}f_n + \Omega_{372}f_{n+\frac{1}{8}} + \Omega_{373}f_{n+\frac{1}{4}} + \Omega_{374}f_{n+\frac{3}{8}} + \Omega_{375}f_{n+\frac{1}{2}} + \Omega_{376}f_{n+\frac{5}{8}} + \Omega_{377}f_{n+\frac{3}{4}} + \Omega_{378}f_{n+\frac{7}{8}} + \Omega_{379}f_{n+1}\right)\\
y'''_{n+1} &= y'''_n + h\left(\Omega_{371}f_n + \Omega_{372}f_{n+\frac{1}{8}} + \Omega_{373}f_{n+\frac{1}{4}} + \Omega_{374}f_{n+\frac{3}{8}} + \Omega_{375}f_{n+\frac{1}{2}} + \Omega_{376}f_{n+\frac{5}{8}} + \Omega_{377}f_{n+\frac{3}{4}} + \Omega_{378}f_{n+\frac{7}{8}} + \Omega_{379}f_{n+1}\right)
\end{aligned}\right\} \quad (11)$$

Hence, in order to obtain the unknown coefficients of Ω, we consider $\psi_{\xi j} = U^{-1}G$ where

$$\begin{pmatrix}\psi_{011}\\ \psi_{012}\\ \psi_{013}\\ \psi_{014}\\ \psi_{015}\\ \psi_{016}\\ \psi_{017}\\ \psi_{018}\\ \psi_{019}\end{pmatrix}=\begin{pmatrix}\frac{24396497}{3923981107200}\\ \frac{1520909}{163499212800}\\ -\frac{13220819}{980995276800}\\ \frac{8390797}{490497638400}\\ -\frac{2050007}{130799370240}\\ \frac{4854761}{490497638400}\\ -\frac{364589}{89181388800}\\ \frac{162689}{163499212800}\\ -\frac{425111}{3923981107200}\end{pmatrix},\begin{pmatrix}\psi_{021}\\ \psi_{022}\\ \psi_{023}\\ \Omega_{024}\\ \psi_{025}\\ \psi_{026}\\ \psi_{027}\\ \psi_{028}\\ \psi_{029}\end{pmatrix}=\begin{pmatrix}\frac{1035731}{15328051200}\\ \frac{169969}{958003200}\\ -\frac{37379}{182476800}\\ \frac{245837}{958003200}\\ -\frac{357779}{1532805120}\\ \frac{46873}{319334400}\\ -\frac{231691}{3832012800}\\ \frac{14071}{958003200}\\ -\frac{8159}{5109350400}\end{pmatrix},$$

$$\begin{pmatrix}\psi_{031}\\ \psi_{032}\\ \psi_{033}\\ \psi_{034}\\ \psi_{035}\\ \psi_{036}\\ \psi_{037}\\ \psi_{038}\\ \psi_{039}\end{pmatrix}=\begin{pmatrix}\frac{4104531}{16148070400}\\ \frac{156411}{183500800}\\ -\frac{3119229}{4037017600}\\ \frac{59337}{57671680}\\ -\frac{1516887}{1614807040}\\ \frac{1194183}{2018508800}\\ -\frac{984537}{4037017600}\\ \frac{17091}{288358400}\\ -\frac{20817}{3229614080}\end{pmatrix},\begin{pmatrix}\psi_{041}\\ \psi_{042}\\ \psi_{043}\\ \psi_{044}\\ \psi_{045}\\ \psi_{046}\\ \psi_{047}\\ \psi_{048}\\ \psi_{049}\end{pmatrix}=\begin{pmatrix}\frac{9521}{14968800}\\ \frac{1511}{623700}\\ -\frac{5239}{2993760}\\ \frac{5011}{1871100}\\ -\frac{25}{10368}\\ \frac{2843}{1871100}\\ -\frac{9379}{14968800}\\ \frac{19}{124740}\\ -\frac{31}{1871100}\end{pmatrix},$$

$$\begin{pmatrix}\psi_{051}\\ \psi_{052}\\ \psi_{053}\\ \psi_{054}\\ \psi_{055}\\ \psi_{056}\\ \psi_{057}\\ \psi_{058}\\ \psi_{059}\end{pmatrix}=\begin{pmatrix}\frac{201421625}{156959244288}\\ \frac{103514375}{19619905536}\\ -\frac{40900625}{13079937024}\\ \frac{111998125}{19619905536}\\ -\frac{383340625}{78479622144}\\ \frac{2901125}{934281216}\\ -\frac{50269375}{39239811072}\\ \frac{79375}{254803968}\\ -\frac{1773125}{52319748096}\end{pmatrix},\begin{pmatrix}\psi_{061}\\ \psi_{062}\\ \psi_{063}\\ \psi_{064}\\ \psi_{065}\\ \psi_{066}\\ \psi_{067}\\ \psi_{068}\\ \psi_{069}\end{pmatrix}=\begin{pmatrix}\frac{142929}{63078400}\\ \frac{38637}{3942400}\\ -\frac{77193}{15769600}\\ \frac{42057}{3942400}\\ -\frac{53217}{6307840}\\ \frac{21951}{3942400}\\ -\frac{5139}{2252800}\\ \frac{2187}{3942400}\\ -\frac{3807}{63078400}\end{pmatrix},$$

$$\begin{pmatrix}\psi_{071}\\ \psi_{072}\\ \psi_{073}\\ \psi_{074}\\ \psi_{075}\\ \psi_{076}\\ \psi_{077}\\ \psi_{078}\\ \psi_{079}\end{pmatrix}=\begin{pmatrix}\frac{2048300303}{560568729600}\\ \frac{382678583}{23357030400}\\ -\frac{89766187}{12740198400}\\ \frac{1268609167}{70071091200}\\ -\frac{244827569}{18685624320}\\ \frac{650716619}{70071091200}\\ -\frac{512596693}{140142182400}\\ \frac{21001547}{23357030400}\\ -\frac{54841241}{560568729600}\end{pmatrix},\begin{pmatrix}\psi_{081}\\ \psi_{082}\\ \psi_{083}\\ \psi_{084}\\ \psi_{085}\\ \psi_{086}\\ \psi_{087}\\ \psi_{088}\\ \psi_{089}\end{pmatrix}=\begin{pmatrix}\frac{2581}{467775}\\ \frac{11888}{467775}\\ -\frac{1492}{155925}\\ \frac{2672}{93555}\\ -\frac{1769}{93555}\\ \frac{208}{14175}\\ -\frac{2468}{467775}\\ \frac{656}{467775}\\ -\frac{37}{249480}\end{pmatrix}$$

Likewise, the unknown coefficients of Ω is given by $\Omega_{\xi j\varsigma}=U^{-1}D$ where

$$\begin{pmatrix}\Omega_{111}\\\Omega_{112}\\\Omega_{113}\\\Omega_{114}\\\Omega_{115}\\\Omega_{116}\\\Omega_{117}\\\Omega_{118}\\\Omega_{119}\end{pmatrix}=\begin{pmatrix}\frac{517129}{2919628800}\\\frac{1129981}{3406233600}\\-\frac{1871827}{4087480320}\\\frac{5887073}{10218700800}\\-\frac{716363}{1362493440}\\\frac{3385541}{10218700800}\\-\frac{2792861}{20437401600}\\\frac{22637}{681246720}\\-\frac{36943}{10218700800}\end{pmatrix},\quad\begin{pmatrix}\Omega_{121}\\\Omega_{122}\\\Omega_{123}\\\Omega_{124}\\\Omega_{125}\\\Omega_{126}\\\Omega_{127}\\\Omega_{128}\\\Omega_{129}\end{pmatrix}=\begin{pmatrix}\frac{286967}{319334400}\\\frac{32543}{11404800}\\-\frac{22063}{7603200}\\\frac{58657}{15966720}\\-\frac{21359}{6386688}\\\frac{55969}{26611200}\\-\frac{138317}{159667200}\\\frac{16799}{79833600}\\-\frac{487}{21288960}\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{131}\\\Omega_{132}\\\Omega_{133}\\\Omega_{134}\\\Omega_{135}\\\Omega_{136}\\\Omega_{137}\\\Omega_{138}\\\Omega_{139}\end{pmatrix}=\begin{pmatrix}\frac{68769}{31539200}\\\frac{1067877}{126156800}\\-\frac{1563651}{252313600}\\\frac{32841}{3604480}\\-\frac{419931}{50462720}\\\frac{661959}{126156800}\\-\frac{546129}{252313600}\\\frac{66393}{126156800}\\-\frac{2889}{50462720}\end{pmatrix},\quad\begin{pmatrix}\Omega_{141}\\\Omega_{142}\\\Omega_{143}\\\Omega_{144}\\\Omega_{145}\\\Omega_{146}\\\Omega_{147}\\\Omega_{148}\\\Omega_{149}\end{pmatrix}=\begin{pmatrix}\frac{80293}{19958400}\\\frac{3571}{207900}\\-\frac{4703}{498960}\\\frac{11213}{623700}\\-\frac{37}{2376}\\\frac{6131}{623700}\\-\frac{10117}{2494800}\\\frac{41}{41580}\\-\frac{2141}{19958400}\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{151}\\\Omega_{152}\\\Omega_{153}\\\Omega_{154}\\\Omega_{155}\\\Omega_{156}\\\Omega_{157}\\\Omega_{158}\\\Omega_{159}\end{pmatrix}=\begin{pmatrix}\frac{5253125}{817496064}\\\frac{11851375}{408748032}\\-\frac{3432125}{272498688}\\\frac{12768625}{408748032}\\-\frac{19680625}{817496064}\\\frac{307625}{19464192}\\-\frac{5327125}{817496064}\\\frac{647875}{408748032}\\-\frac{5875}{34062336}\end{pmatrix},\quad\begin{pmatrix}\Omega_{161}\\\Omega_{162}\\\Omega_{163}\\\Omega_{164}\\\Omega_{165}\\\Omega_{166}\\\Omega_{167}\\\Omega_{168}\\\Omega_{169}\end{pmatrix}=\begin{pmatrix}\frac{37017}{3942400}\\\frac{6183}{140800}\\-\frac{6183}{394240}\\\frac{48141}{985600}\\-\frac{12933}{394240}\\\frac{23787}{985600}\\-\frac{2691}{281600}\\\frac{459}{197120}\\-\frac{999}{3942400}\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{171}\\\Omega_{172}\\\Omega_{173}\\\Omega_{174}\\\Omega_{175}\\\Omega_{176}\\\Omega_{177}\\\Omega_{178}\\\Omega_{179}\end{pmatrix}=\begin{pmatrix}\frac{18850937}{1459814400}\\\frac{30139753}{486604800}\\-\frac{54639557}{2919628800}\\\frac{20689417}{291962880}\\-\frac{1630279}{38928384}\\\frac{52421033}{1459814400}\\-\frac{35782103}{2919628800}\\\frac{1570597}{486604800}\\-\frac{40817}{116785152}\end{pmatrix},\quad\begin{pmatrix}\Omega_{181}\\\Omega_{182}\\\Omega_{183}\\\Omega_{184}\\\Omega_{185}\\\Omega_{186}\\\Omega_{187}\\\Omega_{188}\\\Omega_{189}\end{pmatrix}=\begin{pmatrix}\frac{3029}{178200}\\\frac{12952}{155925}\\-\frac{1126}{51975}\\\frac{3032}{31185}\\-\frac{3191}{62370}\\\frac{2648}{51975}\\-\frac{2102}{155925}\\\frac{808}{155925}\\-\frac{37}{83160}\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{211}\\\Omega_{212}\\\Omega_{213}\\\Omega_{214}\\\Omega_{215}\\\Omega_{216}\\\Omega_{217}\\\Omega_{218}\\\Omega_{219}\end{pmatrix}=\begin{pmatrix}\frac{324901}{92897280}\\\frac{8183}{921600}\\-\frac{653203}{58060800}\\\frac{50689}{3628800}\\-\frac{196277}{15482880}\\\frac{92473}{11612160}\\-\frac{95167}{29030400}\\\frac{7703}{9676800}\\-\frac{5741}{66355200}\end{pmatrix},\begin{pmatrix}\Omega_{221}\\\Omega_{222}\\\Omega_{223}\\\Omega_{224}\\\Omega_{225}\\\Omega_{226}\\\Omega_{227}\\\Omega_{218}\\\Omega_{229}\end{pmatrix}=\begin{pmatrix}\frac{58193}{7257600}\\\frac{3673}{113400}\\-\frac{81}{3200}\\\frac{7729}{226800}\\-\frac{22703}{725760}\\\frac{373}{18900}\\-\frac{14773}{1814400}\\\frac{449}{226800}\\-\frac{521}{2419200}\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{231}\\\Omega_{232}\\\Omega_{233}\\\Omega_{234}\\\Omega_{235}\\\Omega_{236}\\\Omega_{237}\\\Omega_{238}\\\Omega_{239}\end{pmatrix}=\begin{pmatrix}\frac{71661}{5734400}\\\frac{1467}{25600}\\-\frac{4707}{179200}\\\frac{225}{4096}\\-\frac{28143}{573440}\\\frac{11079}{358400}\\-\frac{9141}{716800}\\\frac{2223}{716800}\\-\frac{387}{1146880}\end{pmatrix},\begin{pmatrix}\Omega_{241}\\\Omega_{242}\\\Omega_{243}\\\Omega_{244}\\\Omega_{245}\\\Omega_{246}\\\Omega_{247}\\\Omega_{248}\\\Omega_{249}\end{pmatrix}=\begin{pmatrix}\frac{7703}{453600}\\\frac{388}{4725}\\-\frac{29}{222}\\\frac{1252}{14175}\\-\frac{47}{720}\\\frac{596}{14175}\\-\frac{493}{28350}\\\frac{4}{945}\\-\frac{209}{453600}\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{251}\\\Omega_{252}\\\Omega_{253}\\\Omega_{254}\\\Omega_{255}\\\Omega_{256}\\\Omega_{257}\\\Omega_{258}\\\Omega_{259}\end{pmatrix}=\begin{pmatrix}\frac{56975}{2654208}\\\frac{248375}{2322432}\\-\frac{19375}{774144}\\\frac{143375}{1161216}\\-\frac{641875}{9289728}\\\frac{225}{4096}\\-\frac{12875}{580608}\\\frac{3125}{580608}\\-\frac{3625}{6193152}\end{pmatrix},\begin{pmatrix}\Omega_{261}\\\Omega_{262}\\\Omega_{263}\\\Omega_{264}\\\Omega_{265}\\\Omega_{266}\\\Omega_{267}\\\Omega_{268}\\\Omega_{269}\end{pmatrix}=\begin{pmatrix}\frac{93}{3584}\\\frac{369}{2800}\\-\frac{549}{22400}\\\frac{111}{700}\\-\frac{639}{8960}\\\frac{9}{112}\\-\frac{81}{3200}\\\frac{9}{1400}\\-\frac{9}{12800}\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{271}\\\Omega_{272}\\\Omega_{273}\\\Omega_{274}\\\Omega_{275}\\\Omega_{276}\\\Omega_{277}\\\Omega_{278}\\\Omega_{279}\end{pmatrix}=\begin{pmatrix}\frac{2019731}{66355200}\\\frac{216433}{1382400}\\-\frac{98441}{4147200}\\\frac{1601467}{8294400}\\-\frac{160867}{2211840}\\\frac{55223}{518400}\\-\frac{127253}{8294400}\\\frac{8183}{921600}\\-\frac{57281}{66355200}\end{pmatrix},\begin{pmatrix}\Omega_{281}\\\Omega_{282}\\\Omega_{283}\\\Omega_{284}\\\Omega_{285}\\\Omega_{286}\\\Omega_{287}\\\Omega_{288}\\\Omega_{289}\end{pmatrix}=\begin{pmatrix}\frac{989}{28350}\\\frac{368}{2025}\\-\frac{116}{4725}\\\frac{656}{2835}\\-\frac{227}{2835}\\\frac{656}{4725}\\-\frac{116}{14175}\\\frac{368}{14175}\\0\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{311}\\ \Omega_{312}\\ \Omega_{313}\\ \Omega_{314}\\ \Omega_{315}\\ \Omega_{316}\\ \Omega_{317}\\ \Omega_{318}\\ \Omega_{319}\end{pmatrix}=\begin{pmatrix}\frac{1070017}{29030400}\\ \frac{2233547}{14515200}\\ -\frac{2302297}{14515200}\\ \frac{2797679}{14515200}\\ -\frac{31457}{181440}\\ \frac{1573169}{14515200}\\ -\frac{645607}{14515200}\\ \frac{156437}{14515200}\\ -\frac{33953}{29030400}\end{pmatrix},\ \begin{pmatrix}\Omega_{321}\\ \Omega_{322}\\ \Omega_{323}\\ \Omega_{324}\\ \Omega_{325}\\ \Omega_{326}\\ \Omega_{327}\\ \Omega_{328}\\ \Omega_{329}\end{pmatrix}=\begin{pmatrix}\frac{32377}{907200}\\ \frac{22823}{113400}\\ -\frac{21247}{453600}\\ \frac{15011}{113400}\\ -\frac{2903}{22680}\\ \frac{9341}{113400}\\ -\frac{15577}{453600}\\ \frac{953}{113400}\\ -\frac{119}{129600}\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{331}\\ \Omega_{332}\\ \Omega_{333}\\ \Omega_{334}\\ \Omega_{335}\\ \Omega_{336}\\ \Omega_{337}\\ \Omega_{338}\\ \Omega_{339}\end{pmatrix}=\begin{pmatrix}\frac{12881}{358400}\\ \frac{35451}{179200}\\ \frac{1719}{179200}\\ \frac{39967}{179200}\\ -\frac{351}{2240}\\ \frac{17217}{179200}\\ -\frac{7031}{179200}\\ \frac{243}{25600}\\ -\frac{369}{358400}\end{pmatrix},\ \begin{pmatrix}\Omega_{341}\\ \Omega_{342}\\ \Omega_{343}\\ \Omega_{344}\\ \Omega_{345}\\ \Omega_{346}\\ \Omega_{347}\\ \Omega_{348}\\ \Omega_{349}\end{pmatrix}=\begin{pmatrix}\frac{4063}{113400}\\ \frac{2822}{14175}\\ \frac{61}{28350}\\ \frac{4094}{14175}\\ -\frac{227}{2835}\\ \frac{1154}{14175}\\ -\frac{989}{28350}\\ \frac{122}{14175}\\ -\frac{107}{113400}\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{351}\\ \Omega_{352}\\ \Omega_{353}\\ \Omega_{354}\\ \Omega_{355}\\ \Omega_{356}\\ \Omega_{357}\\ \Omega_{358}\\ \Omega_{359}\end{pmatrix}=\begin{pmatrix}\frac{41705}{1161216}\\ \frac{115075}{580608}\\ -\frac{3775}{580608}\\ \frac{159175}{580608}\\ -\frac{125}{36288}\\ \frac{85465}{580608}\\ -\frac{24575}{580608}\\ \frac{5725}{580608}\\ -\frac{175}{165888}\end{pmatrix},\ \begin{pmatrix}\Omega_{361}\\ \Omega_{362}\\ \Omega_{363}\\ \Omega_{364}\\ \Omega_{365}\\ \Omega_{366}\\ \Omega_{367}\\ \Omega_{368}\\ \Omega_{369}\end{pmatrix}=\begin{pmatrix}\frac{401}{11200}\\ \frac{279}{1400}\\ \frac{9}{5600}\\ \frac{403}{1400}\\ -\frac{9}{280}\\ \frac{333}{1400}\\ \frac{79}{5600}\\ \frac{9}{1400}\\ -\frac{9}{11200}\end{pmatrix},$$

$$\begin{pmatrix}\Omega_{371}\\ \Omega_{372}\\ \Omega_{373}\\ \Omega_{374}\\ \Omega_{375}\\ \Omega_{376}\\ \Omega_{377}\\ \Omega_{378}\\ \Omega_{379}\end{pmatrix}=\begin{pmatrix}\frac{149527}{4147200}\\ \frac{408317}{2073600}\\ \frac{24353}{2073600}\\ \frac{542969}{2073600}\\ \frac{343}{25920}\\ \frac{368039}{2073600}\\ \frac{261023}{2073600}\\ \frac{111587}{2073600}\\ -\frac{8183}{4147200}\end{pmatrix},\ \begin{pmatrix}\Omega_{381}\\ \Omega_{382}\\ \Omega_{383}\\ \Omega_{384}\\ \Omega_{385}\\ \Omega_{386}\\ \Omega_{387}\\ \Omega_{388}\\ \Omega_{389}\end{pmatrix}=\begin{pmatrix}\frac{989}{28350}\\ \frac{2944}{14175}\\ -\frac{464}{14175}\\ \frac{5248}{14175}\\ -\frac{454}{2835}\\ \frac{5248}{14175}\\ -\frac{464}{14175}\\ \frac{2944}{14175}\\ -\frac{989}{28350}\end{pmatrix}$$

3. THE NECESSARY AND SUFFICIENT CONDITIONS FOR ANALYSIS OF THE NEW METHOD

The sufficient and necessary conditions for analysis were basically scrutinized in this section.

3.1 Order and Error Constant of the New Method

We consider the linear operator $L[y(t_n);h]$ with the corollary 2 and 3 below to determining the order and error constant of the new method.

Corollary 2

According to Ref. [28], the linear operator $L[y(t_n);h]$ associate with the local truncation error of the new method is $C_{07}h^{07}y^{07}(t_n)+0(h^{11})$.

Proof

According to Ref. [28], the linear difference operators associated with the new method are given by

$$\left.\begin{array}{l}
L[y(t_n);h]=y\left(t_n+\frac{1}{8}h\right)-\begin{pmatrix}\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\\ \alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_\xi(t)f_{n+\xi}\right)\end{pmatrix}\\
L[y(t_n);h]=y\left(t_n+\frac{1}{4}h\right)-\begin{pmatrix}\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\\ \alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_\xi(t)f_{n+\xi}\right)\end{pmatrix}\\
L[y(t_n);h]=y\left(t_n+\frac{3}{8}h\right)-\begin{pmatrix}\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\\ \alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_\xi(t)f_{n+\xi}\right)\end{pmatrix}\\
L[y(t_n);h]=y\left(t_n+\frac{1}{2}h\right)-\begin{pmatrix}\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\\ \alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_\xi(t)f_{n+\xi}\right)\end{pmatrix}\\
L[y(t_n);h]=y\left(t_n+\frac{5}{8}h\right)-\begin{pmatrix}\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\\ \alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_\xi(t)f_{n+\xi}\right)\end{pmatrix}\\
L[y(t_n);h]=y\left(t_n+\frac{3}{4}h\right)-\begin{pmatrix}\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\\ \alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_\xi(t)f_{n+\xi}\right)\end{pmatrix}\\
L[y(t_n);h]=y\left(t_n+\frac{7}{8}h\right)-\begin{pmatrix}\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\\ \alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_\xi(t)f_{n+\xi}\right)\end{pmatrix}\\
L[y(t_n);h]=y(t_n+h)-\begin{pmatrix}\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\\ \alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_\xi(t)f_{n+\xi}\right)\end{pmatrix}
\end{array}\right\} \quad (12)$$

Corollary 2

According to Ref. [28], the local truncation error of the new method is assume $y(t)$ to be sufficiently differentiable and expanding $y(t_n + qh)$ and $y(t_n + jh)$ about t_n using Taylor series to have

$$L_{\frac{1}{8}}[y(t_n);h] = (-1.0827 \times 10^{-10}),\ L_{\frac{1}{4}}[y(t_n);h] = (-1.6013 \times 10^{-09}),$$
$$L_{\frac{3}{8}}[y(t_n);h] = (-6.45917 \times 10^{-09}),\ L_{\frac{1}{2}}[y(t_n);h] = (-1.6595 \times 10^{-08}),$$
$$L_{\frac{5}{8}}[y(t_n);h] = (-3.3929 \times 10^{-08}),\ L_{\frac{3}{4}}[y(t_n);h] = (-6.0416 \times 10^{-08}),$$
$$L_{\frac{7}{8}}[y(t_n);h] = (-9.6874 \times 10^{-08}),\ L_{1}[y(t_n);h] = (-1.3974 \times 10^{-07})$$

Proof

Expand equation (12) using corollary 2 and then collect the like terms to the power of h gives

$$L_{\frac{1}{8}}[y(t_n);h] = (-1.0827 \times 10^{-10})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{1}{4}}[y(t_n);h] = (-1.6013 \times 10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{3}{8}}[y(t_n);h] = (-6.45917 \times 10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{1}{2}}[y(t_n);h] = (-1.6595 \times 10^{-08})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{5}{8}}[y(t_n);h] = (-3.3929 \times 10^{-08})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{3}{4}}[y(t_n);h] = (-6.0416 \times 10^{-08})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{7}{8}}[y(t_n);h] = (-9.6874 \times 10^{-08})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{1}[y(t_n);h] = (-1.3974 \times 10^{-07})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$

3.2 Consistency

According to Ref. [28], a linear multistep method is said to be consistent if it has an order of convergence greater than or equal to zero i.e. $(p \geq 1)$. Thus, our new schemes are consistent, since the orders are 5.

3.3 Zero Stability

A linear multistep method is said to be Zero-stable for any well behaved initial value problem provided if

i. all roots of $\rho(r)$ lies in the unit disk, $|r| \leq 1$

ii. any roots on the unit circle $(|r| \leq 1)$ are simple

Hence

$$\rho(z) = z^8 - \frac{1522}{35}z^7 + \frac{118124}{105}z^6 - \frac{102528}{5}z^5 + 273664z^4 - 2654208z^3 + 17891328z^2 + 7549742z + 150994944 \quad (13)$$

Now set (12) equal to zero and solving for z gives $z=1$, hence the method is zero stable.

3.4 Convergence

According to Ref. [28], the necessary and sufficient condition for a linear multistep to be convergent is that, it must be consistent and zero stable. Since the new scheme is consistent and zero stable, hence it is convergent.

3.5 Linear Stability

The region of absolute stability of new scheme is the set of complex values λh for which all solutions of the test problem $y'''' = -\lambda^4 y$ will remain bounded as $n \to \infty$ [28].

The concept of A-stability according Ref. [21] is obtained by applying the test equation

$$y^{(k)} = \lambda^{(k)} y \quad (14)$$

to yield

$$Y_m = \mu(z) Y_{m-1},\ z = \lambda h \quad (15)$$

where $\mu(z)$ is the amplification matrix of the form

where $\mu(z)$ is the amplification matrix given by

$$\mu(z) = (\xi^0 - z\eta^{(0)} - z^4\eta^{(0)})^{-1}(\xi^1 - z\eta^{(1)} - z^4\eta^{(1)}) \quad (16)$$

The matrix $\mu(z)$ has Eigen values $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function for of the method is given by

$$\zeta = -\frac{\begin{pmatrix} 1315895675\ 85z^8 - 6937972638\ 972z^7 + 16884257919\ 9360z^6 \\ -32999408631\ 74816z^5 + 4262645178\ 4068096z^4 \\ -4300268967\ 04677888z^3 - 2838814659\ 35100288z^2 \\ -1227258005\ 1546931200\ z + 2416373296\ 7 \end{pmatrix}}{\begin{pmatrix} 8001504000\ z^8 - 3479511168\ 000z^7 + 9001615792\ 000z^6 \\ -1640756404\ 224000z^5 + 2189723590\ 6560000z^4 \\ -212\ 3765592883\ 20000z^3 + 1431575325\ 573120000^2 \\ -6040933241\ 978880000z + 1208186648\ 3957760000 \end{pmatrix}} \quad (17)$$

4. EXPERIMENTAL PROBLEMS AND DISCUSSION

The accuracy of the multi-order schemes was tested on several second, third, and fourth-order oscillatory differential equations, as represented by forms (2), (3), and (4) from physical problems, as well as linear and nonlinear systems. The new method was directly solved based on experiences with the reduction method. All computations were performed using the Maple 18 software package. The absolute errors from the new method were compared with those from existing methods, with variations in step size.

Problem 1: The mass of an object is consider in a dynamic motion that is coined into linear oscillatory form of differential equation (2).

A mass of $10kg$ is attached to a spring having a constant spring of $140\ N/M$. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(v)=5\sin v$. Find the subsequent motion of the mass $(v: 0.10 \le v \le 1.00)$ if the force due to air resistance is $90\left(\frac{du}{dv}\right)N$.

We apply the same procedure, where $m=10,\ k=140,\ a=90$ and $F(v)=5\sin v$ problem 1 reduces to

$$dsolver\left(\left\{\frac{d^2u}{dv^2}+9\frac{du}{dv}+14y(u)=\frac{1}{2}\sin(v),\ u(0)=0,\ u'(0)=-1\right\}\right) \tag{18}$$

with the exact solution of (19) is given by,

$$u(v)=\frac{1}{500}(-90\exp(-2v)+99\exp(-7v)+13\sin v-9\cos v) \tag{19}$$

Source: See Ref. [12,29 and 30].

Problem 2: The highly stiff second oscillatory differential equation of the form

$$u''=u',\ \ u(0)=0,\ u'(0)=-1,\ h=0.1 \tag{20}$$

is consider, whose exact solution is given by

$$u(v)=1-\exp(v) \tag{21}$$

Source: See Ref. [31-33].

Problem 3: Consider the third order oscillatory differential equation

$$u'''+4u'-v=0,\ u(0)=u'(0)=0,\ u''(0)=-1,\ h=0.1 \tag{22}$$

With the exact solution given by

$$u(v)=\frac{3}{16}(1-\cos 2v)+\frac{v^2}{8} \tag{23}$$

Source: See Ref. [34-36].

Problem 4: Consider the highly non-stiff third order Oscillatory problem

$$u'''(v)=3\cos(v),\quad u(0)=1,\ u'(0)=0,\ u''(0)=2 \tag{24}$$

with the exact solution:

$$u(v)=v^2-3\sin(v)+3v+1 \tag{25}$$

Source: See Ref. [37-39]

Problem 5: Consider the highly stiff system of fourth order oscillatory problem

$$u^{iv}=4u'',\ u(0)=1,\ u'(0)=3,\ u''(0)=0,\ u'''(0)=16 \tag{26}$$

with exact solution given by

$$u(v)=1-v+2\exp(2v)-2\exp(-2v) \tag{27}$$

Source: See Ref. [40, 41]

Problem 6: Consider the highly stiff system of fourth order oscillatory problem

$$u^{iv}=\frac{-\left(8+25v+30v^2+12v^3+v^4\right)}{\left(1+v^2\right)},\qquad u(0)=0,\ u'(0)=1,\ u''(0)=0,\ u'''(0)=-3 \tag{28}$$

with exact solution given by

$$u(v)=u\left(1-v^2\right)\exp(v) \tag{29}$$

Source: See Ref. [42, 43].

The following abbreviations are used in the tables and figures below.

ES: Exact Solution
CS: Computed Solution
ENM: Error in New Method
EEM: Error in Existing Method
E[12]: Error in Ref. [12]
E[29]: Error in Ref. [29]
E[30]: Error in Ref. [30]
E[31]: Error in Ref. [31]
E[32]: Error in Ref. [32]
E[33]: Error in Ref. [33]
E[34]: Error in Ref. [34]
E[35]: Error in Ref. [35]
E[36]: Error in Ref. [36]
E[37]: Error in Ref. [37]
E[38]: Error in Ref. [38]
E[39]: Error in Ref. [39]
E[40]: Error in Ref. [40]
E[41]: Error in Ref. [41]
E[42]: Error in Ref. [42]
E[43]: Error in Ref. [43]

Table 1. Computation of NM with Ref. [12, 29 and 30] when solving problem 1

v	**ES**	**CS**	**ENM**	E[12]	E[29]	E[30]
0.1	-0.06436205154552458248	- 0.06436205154553422486	9.6424(-15)	2.0453(-10)	1.2744(-08)	4.4268(-09)
0.2	-0.08430720522644774945	-0.08430720522643955857	8.1909(-15)	4.8485(-10)	3.0442(-08)	2.2383(-08)
0.3	-0.08405225313390041905	- 0.08405225313389655432	3.8647(-15)	6.6174(-10)	4.1501(-08)	3.5865(-08)
0.4	-0.07529304213333374810	- 0.07529304213333333460	4.1350(-15)	7.2649(-10)	4.5385(-08)	4.2157(-08)
0.5	-0.06357063960355798563	- 0.06357063960355967829	1.6927(-15)	7.1295(-10)	4.4298(-08)	4.2895(-08)
0.6	-0.05142117069384508163	- 0.05142117069384780649	2.7249(-15)	6.5550(-10)	4.0466(-08)	4.0288(-08)
0.7	-0.03993052956438697070	- 0.03993052956439003062	3.0599(-15)	5.7884(-10)	3.5475(-08)	3.6051(-08)
0.8	-0.02949865862803573900	- 0.02949865862803873738	2.9984(-15)	4.9808(-10)	3.0285(-08)	3.1287(-08)
0.9	-0.02021269131259124546	- 0.02021269131259398566	2.7402(-15)	4.2140(-10)	2.5408(-08)	2.6618(-08)
1.0	-0.01202699425403169607	- 0.01202699425403410134	2.4053(-15)	3.5257(-10)	2.1071(-08)	2.2352(-08)

Table 2. Computation of NM with Ref. [31-33] when solving problem 2

v	**ES**	**CS**	**ENM**	**E[31]**	**E[32]**	**E[33]**
0.1	-0.1051709180756476248	-0.1051709180756476248	0.0000(00)	7.5650(-11)	3.2482(-12)	2.8589(-15)
0.2	-0.2214027581601698339	-0.2214027581601698339	0.0000(00)	1.6017(-10)	8.5643(-11)	1.4397(-12)
0.3	-0.3498588075760031040	-0.3498588075760031040	0.0000(00)	1.7600(-10)	3.4401(-10)	5.5914(-11)
0.4	-0.4918246976412703178	-0.4918246976412703178	0.0000(00)	6.0784(-10)	7.4251(-10)	4.7966(-09)
0.5	-0.6487212707001281468	-0.6487212707001281468	0.0000(00)	1.4729(-09)	1.3785(-09)	1.0038(-08)
0.6	-0.8221188003905089749	-0.8221188003905089749	0.0000(00)	2.5336(-09)	2.2193(-09)	1.5902(-08)
0.7	-1.0137527074704765216	-1.0137527074704765216	0.0000(00)	4.7876(-09)	3.3875(-09)	2.8700(-08)
0.8	-1.2255409284924676046	-1.2255409284924676046	0.0000(00)	7.2770(-09)	4.8470(-09)	4.2847(-08)
0.9	-1.4596031111569496638	-1.4596031111569496638	0.0000(00)	7.5650(-11)	3.2482(-12)	5.8579(-08)
1.0	-1.7182818284590452354	-1.7182818284590452354	0.0000(00)	1.6017(-10)	8.5643(-11)	8.4493(-08)

Table 3. Computation of NM with Ref. [34-36] when solving problem 3

v	ES	CS	ENM	E[34]	E[35]	E[36]
0.1	0.00498751665476719416	0.00498751665476719417	1.0000(-20)	8.3209(-13)	2.5521(-12)	2.9700(-08)
0.2	0.01980106362445904698	0.01980106362445904699	1.0000(-20)	3.4752(-12)	3.6421(-12)	1.9880(-07)
0.3	0.04399957220443531927	0.04399957220443531929	2.0000(-20)	7.8178(-12)	4.5313(-12)	6.5080(-07)
0.4	0.07686749199740648358	0.07686749199740648362	4.0000(-20)	1.3681(-11)	1.3406(-12)	1.5480(-06)
0.5	0.11744331764972380299	0.11744331764972380306	7.0000(-20)	2.0825(-11)	3.2855(-12)	3.0620(-06)
0.6	0.16455792103562370419	0.16455792103562370429	1.0000(-19)	2.8962(-11)	4.5913(-12)	5.3625(-06)
0.7	0.21688116070620482401	0.21688116070620482414	1.3000(-19)	3.7764(-11)	5.4732(-12)	8.6068(-06)
0.8	0.27297491043149163616	0.27297491043149163631	1.5000(-19)	4.6879(-11)	1.9652(-12)	1.2926(-05)
0.9	0.33135039275495382287	0.33135039275495382304	1.7000(-19)	5.5941(-11)	2.3453(-12)	1.8118(-05)
1.0	0.39052753185258919756	0.39052753185258919775	1.9000(-19)	6.4592(-11)	2.5559(-12)	2.5129(-05)

Table 4. Computation of NM with Ref. [37-39] when solving problem 4

v	ES	CS	ENM	E[37]	E[38]	E[39]
0.1	1.01049975005951554310	1.01049975005951554310	0.0000(00)	2.4800(-07)	1.9700(-16)	0.0000(00)
0.2	1.04399200761481635360	1.04399200761481635380	0.0000(00)	7.3740(-06)	1.2639(-15)	0.0000(00)
0.3	1.10343938001598127470	1.10343938001598127470	0.0000(00)	6.0542(-05)	4.0627(-15)	6.0000(-19)
0.4	1.19174497307404852500	1.19174497307404852500	0.0000(00)	2.5479(-04)	9.4370(-15)	1.7000(-18)
0.5	1.31172338418739099920	1.31172338418739099930	0.0000(00)	7.7602(-04)	1.8205(-14)	3.7000(-18)
0.6	1.46607257981489392840	1.46607257981489392840	0.0000(00)	1.9261(-03)	3.1152(-14)	6.8000(-18)
0.7	1.65734693828692683900	1.65734693828692683900	0.0000(00)	4.1505(-03)	4.9021(-14)	1.1300(-17)
0.8	1.88793172730143171510	1.88793172730143171520	0.0000(00)	8.3637(-03)	7.2504(-14)	1.7300(-17)
0.9	2.16001927111754983460	2.16001927111754983450	0.0000(00)	1.4774(-02)	1.0224(-13)	2.4900(-17)
1.0	2.47558704557631048000	2.47558704557631048000	0.0000(00)	2.4702(-02)	1.3880(-13)	3.4500(-17)

Table 5. Computation of NM when solving problem 5 with h=0.003125

v	**ES**	**CS**	**ENM**
0.003125	1.00937508138036727920	1.00937508138036727920	0.0000(00)
0.006250	1.01875065104675294860	1.01875065104675294860	0.0000(00)
0.009375	1.02812719730424913310	1.02812719730424913310	0.0000(00)
0.001250	1.03750520849609617210	1.03750520849609617210	0.0000(00)
0.015625	1.04688517302275858900	1.04688517302275858900	0.0000(00)
0.018750	1.05626757936100329750	1.05626757936100329750	0.0000(00)
0.021875	1.06565291608298078600	1.06565291608298078600	0.0000(00)
0.025000	1.07504167187531003060	1.07504167187531003060	0.0000(00)
0.028125	1.08443433555816787740	1.08443433555816787740	0.0000(00)
0.031250	1.09383139610438364350	1.09383139610438364350	0.0000(00)

Table 6. Computation of NM when solving problem 5 with h=0.025

v	**ES**	**CS**	**ENM**
0.025	1.07504167187531003060	1.07504167187531003060	0.00000(00)
0.050	1.15033350003968805160	1.15033350003968805160	0.00000(00)
0.075	1.22612626630322531540	1.22612626630322531540	0.00000(00)
0.100	1.30267200508218797520	1.30267200508218797520	0.00000(00)
0.125	1.38022463361633661580	1.38022463361633661580	0.00000(00)
0.150	1.45904058689428523790	1.45904058689428523790	0.00000(00)
0.175	1.53937945887454381400	1.53937945887454381400	0.00000(00)
0.200	1.62150465160563101710	1.62150465160563101710	0.00000(00)
0.225	1.70568403386839551810	1.70568403386839551810	0.00000(00)
0.250	1.79219061098749472320	1.79219061098749472320	0.00000(00)

Table 7. Computation of NM when solving problem 5 with h=0.01

v	ES	CS	ENM
0.01	1.03000266672000050800	1.03000266672000050800	0.00000(00)
0.02	1.06002133504006501740	1.06002133504006501740	0.00000(00)
0.03	1.09007201296111091270	1.09007201296111091270	0.00000(00)
0.04	1.12017072128832277150	1.12017072128832277150	0.00000(00)
0.05	1.15033350003968805160	1.15033350003968805160	0.00000(00)
0.06	1.18057641486221815600	1.18057641486221815600	0.00000(00)
0.07	1.21091556345842144840	1.21091556345842144840	0.00000(00)
0.08	1.24136708202559889650	1.24136708202559889650	0.00000(00)
0.09	1.27194715171053814360	1.27194715171053814360	0.00000(00)
0.1	1.30267200508218797520	1.30267200508218797520	0.00000(00)

Table 8. Computation of NM with Ref. [40, 41] when solving problem 5

v	ENM			EEM	
	$h = 0.003125$	$h = 0.025$	$h = 0.01$	E[40]	E[41]
1	0.0000(00)	0.0000(00)	0.0000(00)	1.0000(-18)	1.00000(-18)
2	0.0000(00)	0.0000(00)	0.0000(00)	2.0000(-18)	2.00000(-18)
3	0.0000(00)	0.0000(00)	0.0000(00)	2.5000(-17)	5.20000(-17)
4	0.0000(00)	0.0000(00)	0.0000(00)	5.3900(-16)	2.39000(-16)
5	0.0000(00)	0.0000(00)	0.0000(00)	5.5200(-16)	5.52000(-16)
6	0.0000(00)	0.0000(00)	0.0000(00)	9.5700(-16)	9.57000(-16)
7	0.0000(00)	0.0000(00)	0.0000(00)	1.2000(-15)	1.20000(-15)
8	0.0000(00)	0.0000(00)	0.0000(00)	6.2700(-15)	1.21000(-15)
9	0.0000(00)	0.0000(00)	0.0000(00)	6.2700(-16)	6.27000(-16)
10	0.0000(00)	0.0000(00)	0.0000(00)	5.5400(-16)	5.54000(-16)

Table 9. Computation of NM with Ref. [42, 43] when solving problem 6

v	ES	CS	ENM	E[42]	E[43]
0.003125	0.00312498470938450965	0.00312498470935964858	2.4861(-14)	2.4874(-14)	1.9902(-14)
0.003125	0.00624987741986711561	0.00624987741907069843	7.9641(-13)	7.9720(-13)	6.3793(-13)
0.009375	0.00937458542869952737	0.00937458542264560338	6.0539(-12)	6.3116(-14)	4.8524(-12)
0.001250	0.01249901526120470559	0.01249901523566763142	2.5537(-11)	4.4102(-12)	2.0482(-11)
0.015625	0.01562307266625029348	0.01562307258823861903	7.8012(-11)	5.7680(-12)	6.2610(-11)
0.018750	0.01874666261169938875	0.01874666241738659932	1.9431(-10)	1.4918(-11)	1.5605(-10)
0.021875	0.02186968927983855277	0.02186968885943323834	4.2041(-10)	9.1931(-11)	3.3786(-10)
0.025000	0.02499205606278295299	0.02499205524232148574	8.2046(-10)	2.7786(-10)	6.5982(-10)
0.028125	0.02811366555785853455	0.02811366407790369113	1.4800(-09)	6.4684(-10)	1.1910(-09)
0.031250	0.03123441956296111601	0.03123441705418685729	2.5088(-09)	1.2977(-09)	2.0204(-09)

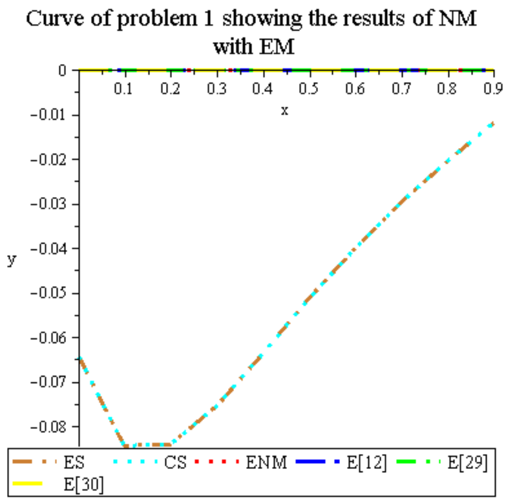

Fig. 1. Graphical curve of Table 1

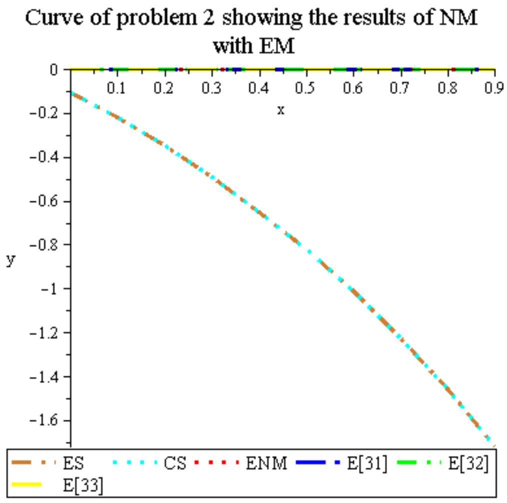

Fig. 2. Graphical curve of Table 2

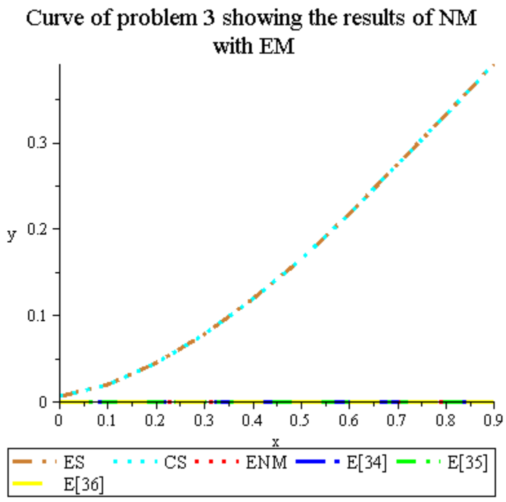

Fig. 3. Graphical curve of Table 3

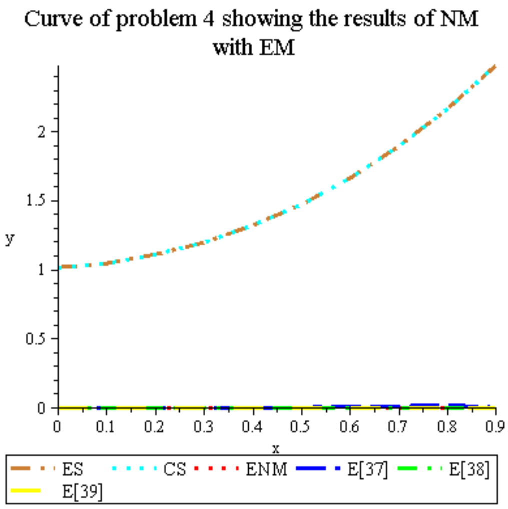

Fig. 4. Graphical curve of Table 4

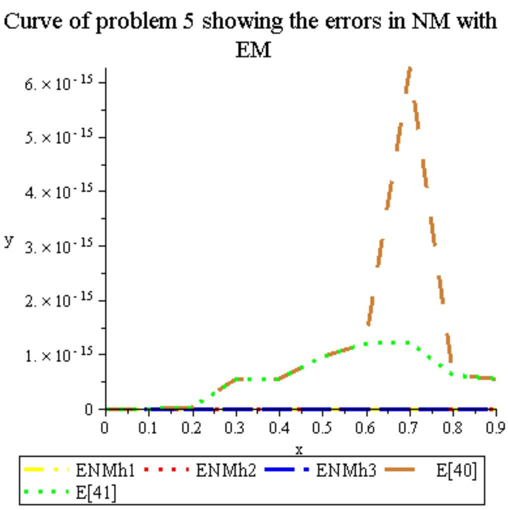

Fig. 5. Graphical curve of Table 8

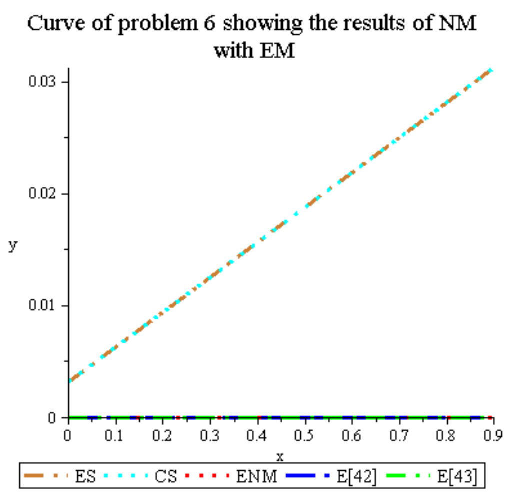

Fig. 6. Graphical curve of Table 9

The direct simulation of higher-order oscillatory differential equation (1) was considered in this study. The new method was directly applied to (2)-(4) to address the limitations of reduction methods.

The oscillatory differential equation (18) in a dynamic system with masses and highly stiff problems (21) were directly handled using the new method (see Problems 1 and 2). Tables 1 and 2 present the results, and these are graphically shown in Figs. 1 and 2. From the results, the new method demonstrates superiority over Ref. [12,29-33].

Additionally, the new method was applied to sampled third-order oscillatory differential equation (22) and non-stiff third order differential equation (24), (see problem 3 and 4), with results shown in Tables 3 and 4 and Figs. 3 and 4. Clearly, the new method performs better than existing methods cited in Ref. [34-39].

Finally, fourth-order linear and non-linear oscillatory problems 5 and 6 were tackled using the new method. The graphical representations of problems 5 and 6 illustrate the effectiveness of the new method compared to Ref. [40-43] (see Tables 5 to 9).

In conclusion, the results presented in Tables 1 to 9 and Figs. 1 to 6 confirm the competence and superiority of the new method over existing methods cited in Ref. [12,29-43] for handling (2)-(4).

5. CONCLUSION

This study delves into the analysis and simulation of second, third, and fourth-order oscillatory systems of higher-order differential equations. The new method adopts a linear block approach in its formulation, and its necessary conditions have been validated. Second, third, and fourth-order problems were directly addressed, demonstrating that the new method offers computational reliability and superiority over other methods used to solve similar oscillatory differential equations.

DISCLAIMER (ARTIFICIAL INTELLIGENCE)

Author(s) hereby declare that generative AI technologies such as Large Language Models, etc have been used during writing or editing of manuscripts. This explanation will include the name, version, model, and source of the generative AI technology and as well as all input prompts provided to the generative AI technology.

Details of the AI usage are given below:

1. ChatGPT 4.0

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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