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Factorization of Modules on Commutative Rings

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Authors' contributions

This work was carried out in collaboration between all authors. Author AK designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Authors MG managed the analyses of the study and managed the literature searches. Both authors read and approved the final manuscript.

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ABSTRACT

In this article, the factorization of a torsion module structure is examined and definitions and theorems related to the uniform factorization part in modules are given. Then, the prime sub-modules of the modules that can be factorized by a single method are examined and basic definitions and theorems are given. We have also studied the module elements in written form as the product of the unreducible elements of the ring with the unreducible elements of the modules with the help of weak prime units defined on the torsion modules.

Keywords: Torsion module; factorization of module; weak prime module.

1. INTRODUCTION

Commutative rings have an important place as algebraic structures. Here are the basics about commutative rings:

For a ring to be commutative, the multiplication of the items in that ring must be commutative. That is $x_1 \cdot x_2 = x_2 \cdot x_1$ all x_1, x_2. It should be valid for x_1, x_2 items. There are two operations in a ring:

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addition (+) and multiplication (·). These processes must provide the following features:

Addition and multiplication are disabled.

Addition and multiplication are combinational.

It is additive, commutative and associative.

For each item there is an additive inverse item.

Multiplication is associative and the distributive property is satisfied (over addition). If there is a unit item (1) in the ring, it is an item that gives itself when multiplied by each item (i.e. a·1=a and 1·a=a). For examples; Integers (Z), is a commutative ring in which "+" and "." operations are performed.

Polynomial Ring: Polynomials over real numbers are a commutative ring in terms of addition and multiplication.

Matrix Ring: n×n matrices, but only 1×1 matrices (i.e., scalars) form a commutative ring. Ideals: In a commutative ring, certain subsets of ideals form substructures compatible with the multiplication and addition structures of the ring.

Fields: If each nonzero item in a commutative ring has an inverse item for multiplication, this ring is called a field. Application Areas Commutative rings have important applications in many mathematical fields such as number theory, algebraic geometry, cryptography.

Rings are generalizing algebraic structures in which the field product does not need to be commutative and multiplicative reciprocals do not need to exist. Informally, a ring is a cluster equipped with two binary transactions that satisfy features similar to the "+" and "." of integers. A ring member can be numbers such as $\mathbb{Z}$ or $\mathbb{C}$, as well as nonnumerical objects such as polynomials, square matrices, functions, and power series. In this paper, R will denote a ring unless otherwise specified (Alan and Özbulur 2016, Anderson et al. 1997, Anderson 1996).

In Atani and Farzilipour 2006, the concept of unique factorization fields (MFUs) generalised to torsion free modules over integral fields, named factorization of modules, and some fundamental theorems for factorization of modules heve been proven.

The aim of this paper is to expand the study of factorization of modules to large families of modules in order that look for farther similarities among the theory of factorization of modules and MFUs. First, the known definitions of factorization of modules and related literature are given. The fundamental theorem of factorization of modules is proved, which states 5 cases, each of which is equal to a module being factorization of in a MFU. We support the notation of principal sub-modules analogous to principal ideals of rings. A module T on a non domain MFU R is proved to be factorization of if and only if T has nonzero principal sub-modules each includes one item of the form pn for some unreducible items z∈T and s ∈ R, respectively. The superposition of a MFU R is in case a factorization of expansion if it is a factorization of module R. In this paper (g.c.d.) will denote the greatest common denominator and (l.c.m.) the least common multiple, unless otherwise stated.

2. UNIFORM FACTORİZATİON İN MODULES

Definition 2.1 Suppose a, b ∈ $\mathbb{Z}$ in which at least one of them is not zero. The largest common divider of a and b, expressed by gcd(a,b), d ∈ $\mathbb{Z}^+$, which provides the next:

i-) q|x , q|b
ii-) If c|x , c|b so, c≤q.

Definition 2.2. An element 0≠t ∈ M is called unreducible element if t has no appropriate factor in M.
It is obvious that, t ≠ 0 for x ∈ R is unreducible ⇔t = xt ' and t'∈ M implies x ∈ U.

Definition 2.3 An element 0≠t ∈ M is said to be primal, if t | 0 ≠ a ∈ R and xt' for t'∈ M, i.e. t|t'.

Proposition 2.1 Let $0 \neq \mathrm{t} \in \mathrm{M}$. So, the next circumtances are tantamount:

1-) t is primal;
2-) Rt cyclic sub-module is clear;
3-) If x ∈ M, so Rx∩Rt =(0) or Rx ⊆ Rt.

Proof. See (Sontag et al., 1978).

Definition 2.4 An item $u \in R$ is termed principal to an R module M if

i-) u is unreducible in R and
ii-) u | x ∈ R and t ∈ M for u | R or x in u | t in M.

Definition 2.5. A torsion free M-module over an integral domain R is named a unique factorization module (MFU) or a factorization of module if the next rules are provided:

[UF1] Each 0≠x ∊ M has an unreducible factorization, i.e. x = a_1 a_2... a_nt, where a_1 a_2... a_n is unreducible in R and m is unreducible in M.

[UF2] If x = a_1 a_2... a_nt = b_1 b_2... b_nt' is an unreducible factorization of x into two factors, then n = k , t ~ t' in M and we can rearrange the order of the b_i's due to $a_i \sim b_i$ in R for each i∊{l,2...,n}.

As stated in Atani and Farzilipour 2006, if an R module M is a MFU, R must mandatory be a MFU. So, when seeking for the factorizability of an R module, let's suppose from the beginning that the R ring is a MFU. If R is a MFU, we state that the next [UFI'] rule means [UFI].

3. FACTORİZATİON OF MODULES

In this section, we will examine the following basic characterization of the factorization of module and its implementation to some patterns.

Theorem 3.1 Let M be a nonzero module on a MFU R that providies the [UFI] rule. That is, the next rules are equivalence:

1-) M is factorization of over R;
2) Each item unreducible of M is primal;
3) for some double of items a ∊ R and h ∊ M, g.c.d. {a,h) ∊ M;
4) for some double of items a ∊ R and h ∊ M, l.c.m {a, h) ∊ M, so the sub-module aM∩ bM is cyclic;
5-) Each item unreducible p of R is prime to M;
6-) (i) $a, b\ \in\ R$ in fact aM ⊆ bM if b|a
(ii) for each pair $a, b\ \in\ R$, there exists an item $c\ \in\ R$ in fact $aM \cap bM\ = cM$ (Atani and Farzilipour 2006).

Remark. If M is a MFU, any item c providing (i) and (ii) is certainly an L.c.m. {a, b} in R.

Proof. $(1) \Leftrightarrow (2)$. Obvious.

$(2) \Rightarrow (3)$: If m = 0 so, g.c.d. {a, m) ~ a for each a ∊ R. If m = bm_0 ≠ 0, where b ∊ R and m_0 is an unreducible item of M, then g.c.d. {a, m} ~ g.c.d. {a, b} = d ∊ R. Obviously, d is a widespread divider of a and m. Suppose d' is other widespread divider of a and m, and write m = bm_0=d'm' for some m' ∊ M. Then d' | b so d'| d, since m_0 is primal and d = g.c.d. {a, b}. Consequently, d~g.c.d. {a,m}; hence (3) holds.

(3) ⇒ (4): If a = 0 then L.c.m. {a, m) ~ 0 ∊ M for each m ∊ M . For 0 ≠ a ∊ R and m ∊ M let d ~ g.c.d. {a, m}. So, for some a' ∊ R and m' ∊ M in fact a = da' and m = dm' in fact g.c.d. {a', m'} ~ 1. It can now be confirmed that m'= a'm is an L.c.m. {a, m}.

((4) ⇒ (5): Let $p\ \in\ R$,p be an unreducible item, in fact p|a ∊ R and m ∊ M.

(5) ⇒ (6): (i) Let $a, b\ \in\ R$ in fact aM ⊆ bM . Assuming b ≠ 0, b | A. Assuming b ≠ 0, b | am_0 for any unreducible item m_0∊ M, hence am_0 for every principal factor p of p|b. By (5), a for every principal factor p of p|b; so, b|A.

(ii) If a = 0 or b = 0, then trivially c = 0. Asserting that a ≠ 0 and b ≠ 0, put c ~ L.c.m. {a, b} and d ~g.c.d. {a, b}. So, aM∩ bm⊇cM and c = a'b = ab', where a' = a|d and b' = b|d If w is any nonzero item of M in fact w = am = bm' ∊ aM∩ bm , then a'm = b'm' | b'm' and (a', b') ~ 1, a' | m' with the just like argument as in the proof of part (i). Eventually, c = a'b divides w = bm'. Accordingly, aM∩ bm⊆cM and (ii) is true.

(d) (6) ⇒ (2) ⇒ (1): If $m \in M$, m is an unreducible item, in fact am ' = bm for any a,b ∊ R and m ' ∊ M in fact am ' = bm ∊ aM∩ bm=cM for any c ∊ R (6), (ii) and (6), (i), b | c and a | C. Because of m is unreducible, a | b and m|m' when b ~ c. So m is primal. Hence (2) is true, and by part (a) above, (2) ⇒ (1).

Our subsequent conclusions provide some fundamental information about the factorization of modules. Most of these have anyway been debated and proven in (Fletcher 1969); but, some of the proofs are uneventually long.

Theorem 3.2 We propose that each module in the consequences is a nonzero module.

Proof. See [13].

Corollary 3.1 Each cyclic module Rm on a MFU Rm is a MFU in which each primal item is a part of M.

Proof. See (Kaplansky 1970)

Corollary 3.2 Each vector space is a (trivial) MFU in which each nonzero vector is primal.

Proof. See [13].

Corollary 3.3. Let K be the quotient space of a MFU R and let M be an R-sub-module of K. In this case, M is factorization of necessary and sufficient condition it is cyclic. Therefore, an ideal of R is a MFU on R necessary and sufficient condition it is a basic ideal.

Proof. For any nonzero pair of items x = a|b and y = c|d of M , we have 0 ≠ bcx= ady ∈ Rx ∩ Ry . M has at most one cyclic sub-module produced by a primal item. It is now easy to see Corollary 3.3.

Corollary 3.4 Each pure sub-module N of a factorization of module M on a MFU R is also a factorization of R-module, and each unreducible element here remains unreducible in M.

Proof. It is obvious that N providies [UFI']. For some a ∈ R and m ∈ M, aN∩Rm= (aM ∩ N) ∩ Rm = (aM ∩Rm) ∩N=Rx∩N=Rx by (4) of Theorem 3.1 for some x∈M and of purity N. Therefore, N is factorization of R. The second statement is clear.

Corollary 3.5 Let {m_i:i∈R} be a class of modules on MFU R. In that case, the next expressions are tantamount:

1-) $P_{(i\in I)}$ is factorization on M_i R;
2-) $\otimes_{(i\in I)}$ is factorization on M_iR;
3-) Each M_i is factorization on R.

Proof. Corollary 3.4 states: (1)⇒ (2)⇒ (3). Let (3) hold and let $\Pi_{(i\in I)} M_i$ = M. If m = $(M_i)_{(i\in I)}$ ∈M, where $m_i = a_i b_i \in M_i$ for some a_i ∈R and an unreducible element m_i', then m is g.c.d-free necessary and sufficient conditions the set of items {a_i | i∈I} in R. So we see that M providies [UFI]. Let p be any unreducible item of R in fact p | I am in M for any a ∈ R and m = $(M_i)_{(i\in I)}$ ∈M, so p | a$m_i \in M_i$ for each i. Consequently, p | m and hence. $\Pi_{(i\in I)} M_i$ = M is again factorization of due to (5) of Theorem 2.1. So (3) ⇒ (1).

Corollary 3.6 Let M be a only factorization Bézout space R, especially a module over a basis ideal space. In this case, (1) M is factorization necessary and sufficient conditions it providies [UFI], and (2) if M is a UFM, so it is a completely flat R-module Atani and Farzilipour).

Proof. (1) The requirement is obvious. To demonstrate proficiency, let p be an unreducible item of R in fact p | am for a ∈ R and m ∈ M . Assume that p ∤ a, then there exist any s and t in R in fact (p,a) = 1 = ps + at. Now, m = psm + atm since p | m; hence M is factorization of by Atani 2006 of Theorem 3.1.

4. PRIME SUB-MODULES OF FACTORIZATION OF MODULES

In this part, we will consider two classes of sub-modules of a factorization of module that play similar roles to the basic principal ideals in a MFU (Costa 1976).

Definition 4.1 A suitable sub-module N of a torsion-free module M on a ring R is named a principal sub-module if x ∈ M means a ∈ R and ax ∈ N means x∈ N or a∈ (N:M). Obviously, each principal ideal F of a ring R is a principal sub-module of the R-module with (P:R)=P. It is also obvious that each torsion-free module includes the principal sub-module (0).

In the next Result 3.1- Result 3.3, the M modules must be unbendable.

Corollary 4.1 If N is a principal sub-module M of an R-module, then (N:M) is a principal ideal of R.
Proof. See (Sharp 2000).

Corollary 4.2 Each maximal sub-module is principal (Sharpe 1987).

Proof. See [13].

Corollary 4.3 A proper sub-module of an M-module N is pure necessary and sufficient conditions it is a principal sub-module with N : M = (0).

Proof. See (Kaplansky 1970).

Proposition 4.1 Let M, R be a module over the completeness region and m ∈ M in fact Rm ≠M. So, m is primal, necessary and sufficient conditions Rm is a principal sub-module with Rm : M = (0).

Proof. It is obvious (Nicolas 1974).

Proposition 4.2 Let M be a module over the integral domain R in fact pM≠ M for each nonunitary item p∈ R. In this case, the following two expressions are equivalent:

1-) p is principal to M;
2-) pM is a principal sub-module of M (pM:M) = (p).

Proof. It is clear to show that (1) implies (2). To prove the opposite, we first notice that, p is unreducible. In the opposite case, we arrive at the PM = M contradiction. Now, the rest of the evidence is clear.

Corollary 4.4 Let M be a cyclic module Rx over a UFD R and $m \in M$ in fact

m ≁ x. So, Rm is a principal sub-module

⇔m ~ px for any irreducible item p of R,
⇔ Rm = pM for any unreducible item p of R.

Proof. if we set m = ax to be a ∈ R, in that case Rm : Rx = (a) becomes. If Rm is principal, in that case (a) the result is a principal ideal with respect to 3.1. Therefore, for an unreducible item p ∈ R, m ~ px.

Definition 4.2 In a torsion-free module, a nonzero principal sub-module is said to be no fewer than if it does not contain a suitable principal sub-module other than (0).

Theorem 4.1. Let N be a nonzero sub-module of a factorization of R-module M. So,

(1) N is a no fewer than principal sub-module with N : M = (0) necessary and sufficient conditions N = $R\eta \underset{\neq}{\subset} M$ for a primal item η of M.

(2) N is a no fewer than principal sub-module with $N : M \neq (0)$ necessary and sufficient conditions N = pM for an unreducible item $p \in R$.

Proof. (1) If N is a no fewer than principal sub-module with N∶ M = (0), then N includes the fundamental item η ∈ M . Since N is no fewer than and Rn is principal, so N = Rn. The converse is easy to see because any nonzero principal sub-module of M included in Rn where Rn is primal is pure and so, includes n.

(2) This requirement follows from the fact that a nonzero principal ideal N : M must include an unreducible item p R in fact pM ⊆ N, where pM is a principal sub-module. To prove qualification, let N' be a principal sub-module of M included in N = pM for an unreducible item of R. We have seen that, N is a principal sub-module such that N∶ M = (p), so N includes no primal item of M . We claim that, (N'∶M) = (p); or else, (N'∶M) = (0) and N' ⊆ N includes a primal item. It is now clear that, pM = N = N'. Hence, N = pM is a no fewer than principal sub-module such that (N∶M) = (p) ≠ (0).

As it is well known, an integer field is a MFU necessary and sufficient conditions each nonzero basic ideal includes a basic ideal (Nicolas 1967).

Theorem 4.2 Let R be a MFU which is not a field. An R-module M is a UFM necessary and sufficient conditions;

(i-) M includes nonzero principal sub-modules
(ii-) each of the nonzero principal sub-modules has an item of the form pη for a fundamental item η∈ M and a principal item p ∈ R.

Proof. This requirement follows from Theorem 3.1 and the fact that each nonzero principal sub-module of a factorization of module includes a no fewer than principal sub-module. Note that, both types of no fewer than principal sub-modules include an item of the form pn, as introduced in the theorem. To prove the proficiency, let S be the class of all primal items in M and put S* = $\{a\eta] \mid \eta \in S \; and \; a \in R - \{0\}\}$; so, S* ≠ ∅ since $S \neq$ ∅ by the hypothesis. We notice that, M is factorization of necessary and sufficient conditions S* = $M - \{0\}$. Let, S* $\underset{\neq}{\subset} M - \{0\}$ and e be a nonzero item of M - S*, so $Re \subseteq M$ – S*. In the contrary case, e ∈ S* ∩ (M – S*) = ∅ , because any primal item divides e in M if Re ∩ S* ≠ ∅ . Then, by exclusion of S*, we extend Re to the sub-module N of the maximum of M; such N must exist by Zorn's Lemma. We claim that N is pure. If this is not the case, so there exists an O≠a ∈ R and m ∈ M - N in fact am ∈ aM ∩ N. Because of (N + Rm) ∩ S* ≠ 0, there exists an element w ∈ S* in fact w = x + rm for some x ∈ N and r ∈ R, implying the discrepancy aw = ax + ram ∈ N∩ S* = ∅. Therefore, since N is pure, it is principal. Because of the hypothesis, we are faced with the contradiction that N ∩ S* ≠ ∅. Hence, S* = M - {0}, so M is factorization of R.

5. CONCLUSION

In this study, the applications of unidirectional factorizable features in rings in modules were investigated. In later studies, the answer to the question of whether R is a commutative and unitary ring and M is a torsion-free module, while M is a single factorizable module requires R to be a smooth factorizable ring, can be examined.

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REFERENCES

Alan, M., & Özbulur, E. (2016). On unique factorization modules. *International Journal of Pure and Applied Mathematics, 108*(23–28).

Anderson, D. D., & Valdes-Leon, S. (1996). Factorization in commutative rings with zero divisors. *Rocky Mountain Journal of Mathematics, 26*(439–480).

Anderson, D. D., & Valdes-Leon, S. (1997). Factorization in commutative rings with zero divisors, II. In *Factorization in Integral Domains* (pp. 197–219). *Lecture Notes in Pure and Applied Mathematics, 189*.

Atani, S. E. (2006). On graded prime submodules. *Chiang Mai Journal of Science, 33*(1), 3–7.

Atani, S. E., & Farzilipour, F. (2006). Notes on the graded prime submodules. *International Mathematical Forum, 1*(38), 1871–1880.

Costa, D. L. (1976). Unique factorization in modules and symmetric algebras. *Transactions of the American Mathematical Society, 224*(2), 267–280.

Fletcher, C. R. (1969). Unique factorization rings. *Proceedings of the Cambridge Philosophical Society, 65*, 579–583.

Kaplansky, I. (1970). *Commutative rings*. Allyn and Bacon.

Nicolas, A. M. (1967). Modules factoriels. *Séminaire Dubreil-Pisot, 20*(10), 1–12.

Nicolas, A. M. (1974). Extensions factorielles. *Séminaire Dubreil, 27*(15), 1–8.

Sharp, R. Y. (2000). *Steps in commutative algebra* (London Mathematical Society Student Texts, 51). Cambridge University Press.

Sharpe, D. W. (1987). *Rings and factorization*. Cambridge University Press.

Sontag, E.D., Dicks, W. (1978). Sylvester Domains, *J. Pure and Applied Algebra* 13, 243-275.

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