

Research Article

Asymptotic Behavior of Solution for Functional Evolution Equations with Stepanov Forcing Terms

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Through the use of the measure theory, evolution family, “Acquistapace–Terreni” condition, and Hölder inequality, the core objective of this work is to seek to analyze whether there is unique μ -pseudo almost periodic solution to a functional evolution equation with Stepanov forcing terms in a Banach space. Certain adequate conditions are derived guaranteeing there is unique μ -pseudo almost periodic solution to the equation by Lipschitz condition and contraction mapping principle. Finally, an example is used to demonstrate our theoretical findings.

1. Introduction

Periodicity is a common phenomenon in the natural and social sciences that varies over time. Periodicity can be caused by business and economic activities, and it differs from trend changes in that it is not a gradual development in one direction, but rather a series of rising and falling oscillations. Sunrise and sunset, the cycle of the seasons, and the moon changing from deficit to surplus are all cyclical phenomena. In epidemiology, the cycle refers to the frequency of disease according to a certain interval of time, the regular ups and downs of fluctuations; every several years, there is an epidemic peak phenomenon. In physics, it refers to the reciprocal motion of a mass in simple harmonic motion, which is always cyclic, with each cycle taking the same amount of time and being strictly periodic. In mathematics, a function whose output value repeats periodically is called a periodic function. The motion of the planets is not regular and elliptical, because it can be influenced by other stars; the transmission of radio waves can also be disturbed. Thus, the periodicity will no longer be maintained. In such a case, the almost period would better reflect the variation of the above phenomenon with time. In mathematics, the almost periodic function is a class of functions with approximately periodic properties, which is the extension of the continuous periodic function. Different periodic functions have different periods,

and their sums, differences, or products are not necessarily periodic functions anymore, although the almost periodic function may not have strict periodicity, but can have some better properties than the periodic function. In theory, all periodic functions in any paradigm do not constitute a Banach space, while the probable periodic functions in the upper bound paradigm constitute a Banach space. This also shows that the probable periodic function will be more widely used than the periodic function, and it is more practical to discuss the probable periodic solution of the equation.

In fact, strict periodic changes are difficult to appear in nature. In real life, there are not many strict periodic phenomena such as seven days a week. On the contrary, almost periodic changes are more likely to accurately describe the change law of nature. Almost periodic phenomena are more common than periodic phenomena, and almost periodic phenomena are closer to reality than periodic phenomena. In reality, almost periodic phenomena are more likely to occur.

Almost periodic function can better reflect the change law of things in real life and explain the phenomena in real life. It is a phenomenon easier to see in natural science and social science. Almost periodic results are more common than periodic results. Almost periodic function has many other good properties, and its practical application is more and more widely in real life. The almost periodic property

of the solution of the equation has attracted the attention of many scholars. Discussing the almost periodic property of the solution of the equation has more extensive practical significance and practical application value. Therefore, it has become an important topic in mathematical research.

In 1992, Zhang proposed the idea of pseudo almost periodic functions in the academic paper [1]. After that, in 2006, the idea of weighted pseudo almost periodic functions was presented by Diagana in the literature [2]. As an extension of the concept of pseudo almost periodic functions and weighted pseudo almost periodic functions, Blot et al. presented the idea of μ -pseudo almost functions in 2013 [3]. Since then, many scholars have conducted extensive research on the properties of such functions and made advantage of these properties to set up the existence of μ -pseudo almost periodic solutions for partial functional non-autonomous equation [4, 5], partial functional differential equations [6, 7], partial differential equation [8], nonautonomous integrodifferential equations [9], and semilinear fractional integrodifferential equations [10]. In 2007, Diagana proposed the conception of the Stepanov pseudo almost periodic function in the literature [11]. Then, in 2010, Diagana et al. presented the conception of weighted Stepanov-like pseudo almost periodic function in the literature [12], as the generalization of Stepanov pseudo almost periodic functions. In the literature [13], the authors Es-sebbar and Ezzinbi proposed the notion of Stepanov μ -pseudo almost periodic function. In [14], the authors studied whether there is unique μ -pseudo almost periodic solution for a parabolic evolution equation and the equation's forced terms are assumed to be μ -Stepanov pseudo almost periodic. In [15], the authors established the convolution and composition theorems of the μ -Stepanov pseudo almost periodic functions and verified whether there is unique μ -pseudo almost periodic mild solution of fractional integrodifferential equation.

Inspired by the above, in this work, let $(K, \|\cdot\|)$ be a Banach space and $1 < p < \infty$; by the measure theory, we mostly prove whether there is unique μ -pseudo almost periodic solution to a functional evolution equation

$$\begin{aligned} \frac{d}{dt} \left[z(t) - \psi \left(t, \widetilde{C}_1(t)z(t) \right) \right] &= A(t) \left[z(t) - \psi \left(t, \widetilde{C}_1(t)z(t) \right) \right] \\ &\quad + g \left(t, z(t), \widetilde{C}_2(t)z(t) \right), \end{aligned} \quad (1)$$

for $t \in \mathbb{R}$, where $\{A(t)\}_{t \in \mathbb{R}}$ fulfills the Acquistapace-Terreni condition, which is exponentially stable, and $Q(t, s)$ created by $A(t)$ and $\psi : \mathbb{R} \times K \rightarrow K$, is μ -pseudo almost periodic and $g : \mathbb{R} \times K \times K \rightarrow K$ is Stepanov-like μ -pseudo almost periodic functions; $\widetilde{C}_1(t), \widetilde{C}_2(t) (t \in \mathbb{R})$ are families of (perhaps unbounded) linear operators. The topic about whether there is unique μ -pseudo almost periodic solution to functional evolution equation with Stepanov forcing terms of (1) is untreated in the past work, which is one of the crucial incentives of this study.

The following is an outline for this work. Section 2 introduces some essential concepts and basic properties. Section 3 is dedicated to demonstrating whether there are unique μ -pseudo almost periodic solutions of (1). To give you an illustration, in the last part, we provide a case supporting our conclusion.

2. Preliminaries

Throughout the course of this project, $(K, \|\cdot\|)$ and $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ are two Banach spaces. $BC(\mathbb{R}, K)$ (respectively, $BC(\mathbb{R} \times \mathbb{H}, K)$) is the Banach space of bounded continuous functions from \mathbb{R} into K (respectively, jointly bounded continuous functions $\psi : \mathbb{R} \times \mathbb{H} \rightarrow K$) as a result of its norm by

$$\|\psi\|_{\infty} = \sup_{t \in \mathbb{R}} \|\psi(t)\|. \quad (2)$$

We signify by \mathcal{D} the Lebesgue \widehat{D} -field of \mathbb{R} and by \mathcal{N} the set of all positive measures μ on \mathcal{D} such that $\mu(\mathbb{R}) = +\infty$ and $\mu([c, d]) < +\infty$ for all $c, d \in \mathbb{R} (c \leq d)$.

From $\mu \in \mathcal{N}$, the following assumptions are assumed.

(H1) For all c, d , and $e \in \mathbb{R}$, such that $0 \leq c < d \leq e$, there exist $\widehat{v}_0 \geq 0$ and $\alpha_0 \geq 0$ satisfying $|\widehat{v}| \geq \widehat{v}_0 \Rightarrow \mu((c + \widehat{v}, d + \widehat{v})) \geq \alpha_0 \mu([\widehat{v}, e + \widehat{v}])$.

(H2) For all $\widehat{v} \in \mathbb{R}$, there exist $\tilde{\alpha} > 0$ and a bounded interval I satisfying

$$\mu(\{c + \widehat{v} : c \in B\}) \leq \tilde{\alpha} \mu(B), \quad \text{if } B \in \mathcal{D} \text{ such that } B \cap I = \emptyset. \quad (3)$$

Definition 1 (see [16]). Let $\psi(t) \in C(\mathbb{R}, K)$, $\psi(t)$ be referred to as almost periodic if, for any $\varepsilon > 0$, the set $E(\psi, \varepsilon) = \{\widehat{\kappa} : \|\psi(t + \widehat{\kappa}) - \psi(t)\| \text{ for all } t \in \mathbb{R}\}$ is relatively dense; that is, for any $\varepsilon > 0$, it is possible to locate a real number $l = l(\varepsilon) > 0$; for any interval with length $l(\varepsilon)$, there exists a number $\widehat{\kappa} = \widehat{\kappa}(\varepsilon)$ in this interval satisfying $\|\psi(t + \widehat{\kappa}) - \psi(t)\|$ for all $t \in \mathbb{R}$. We denote the space of all such functions by $AP(\mathbb{R}, X)$.

Consider the function F , which is equal to

$$F(x) = \sin(x) + \sin(\sqrt{2}x). \quad (4)$$

F is an almost periodic function, while F is not periodic. See almost periodic function F below in Figure 1.

Definition 2 (see [17]). A continuous function $\psi : \mathbb{R} \times \mathbb{H} \rightarrow K$ is regarded as almost periodic in t uniformly in relation to \tilde{u} in \mathbb{H} , if for any compact subset $K \subset \mathbb{H}$ and every $\varepsilon > 0$, there exists $l > 0$ such that every interval of length l , it is possible to find a number \widehat{v} in this interval with the feature that

$$\|\psi(t + \widehat{v}, \tilde{u}) - \psi(t, \tilde{u})\| < \varepsilon, \quad \text{for } (t, \tilde{u}) \in \mathbb{R} \times K. \quad (5)$$

The union of such functions will be referred to as $APU(\mathbb{R} \times \mathbb{H}; K)$.

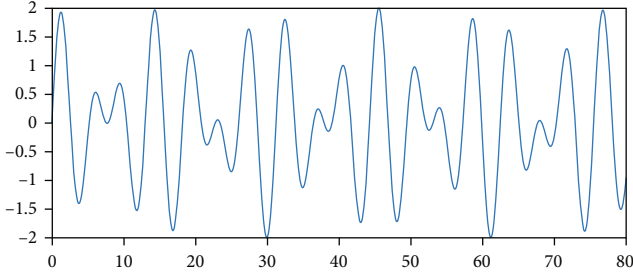


FIGURE 1: Almost periodic function F .

Proposition 3 (see [16]). *If ψ, ψ_1 and $\psi_2 \in AP(K)$, then*

$$\psi_1 + \psi_2 \in AP(K). \quad (6)$$

- (i) $\lambda\psi \in AP(K)$ for any scalar λ
- (ii) $\psi_\alpha \in AP(K)$ where ψ_α is the right translation function of ψ : $\mathbb{R} \rightarrow K$ is characterised by $\psi_\alpha(\cdot) = \psi(\cdot + \alpha)$
- (iii) The range $\mathcal{R}_\psi := \{\psi(t) : t \in \mathbb{R}\}$ is a little package in K ; thus, ψ is bounded in norm
- (iv) If $\psi_n \rightarrow \psi$ uniformly on \mathbb{R} where each $\psi_n \in AP(K)$, then $\psi \in AP(K)$ too

Definition 4 (see [3]). Let $\mu \in \mathcal{N}$. A continuous bounded function $\psi : \mathbb{R} \rightarrow K$ is regarded as μ -ergodic if

$$\lim_{\omega \rightarrow +\infty} \frac{1}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \|\psi(t)\| d\mu(t) = 0, \quad (7)$$

where $\widehat{\omega} := [-\omega, \omega]$. All of these functions' space is denoted by $\mathcal{E}(\mathbb{R}; K, \mu)$.

Definition 5 (see [3]). Let $\mu \in \mathcal{N}$. A continuous function $\psi : \mathbb{R} \times K \rightarrow \mathbb{H}$ is regarded as μ -ergodic in $t \in \mathbb{R}$ uniformly in relation to $x \in K$, if the two following statements are satisfied:

- (i) For all $x \in K$, $\psi(\cdot, x) \in \mathcal{E}(\mathbb{R}; K, \mu)$
- (ii) ψ is uniformly continuous on every compact set K in terms of the second variable x

The union of such functions will be referred to as $\mathcal{E}U(\mathbb{R} \times K; \mathbb{H}, \mu)$.

Proposition 6 (see [13]). *If a sequence $(\psi_n)_n$ of μ -ergodic functions uniformly converges to a function ψ , then ψ is also μ -ergodic.*

Theorem 7 (see [3]). *Let $\mu \in \mathcal{N}$. Then, $(\mathcal{E}(\mathbb{R}; K, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Definition 8 (see [3]). Let $\mu \in \mathcal{N}$. A continuous function $\psi : \mathbb{R} \rightarrow K$ is called μ -pseudo almost periodic if it is able to create it as

$$\psi = \psi_1 + \psi_2, \quad (8)$$

where $\psi_1 \in AP(\mathbb{R}; K)$ and $\psi_2 \in \mathcal{E}(\mathbb{R}; K, \mu)$. Indicate with $PAP(\mathbb{R}; K, \mu)$ the area occupied by such functions.

Definition 9 (see [3]). Let $\mu \in \mathcal{N}$. A continuous function $\psi : \mathbb{R} \times K \rightarrow \mathbb{H}$ is called μ -pseudo almost periodic in $t \in \mathbb{R}$ uniformly in relation to $x \in K$, if it is possible to write it in the form

$$\psi = \psi_1 + \psi_2, \quad (9)$$

where $\psi_1 \in APU(\mathbb{R} \times K; \mathbb{H})$ and $\psi_2 \in \mathcal{E}U(\mathbb{R} \times K; \mathbb{H}, \mu)$. The set of such functions is represented by the symbol $PAPU(\mathbb{R} \times K; \mathbb{H}, \mu)$.

Theorem 10 (see [3]). *Let $\mu \in \mathcal{N}$, $\psi \in PAPU(\mathbb{R} \times \mathbb{H}; K, \mu)$, and $u \in PAP(\mathbb{R}; \mathbb{H}, \mu)$. Assume for all bounded subset B of \mathbb{H} , f is bounded on $\mathbb{R} \times B$. Then, the function*

$$[t \mapsto \psi(t, u(t))] \in PAP(\mathbb{R}; K, \mu). \quad (10)$$

Definition 11 (see [12]). Let $1 \leq p < \infty$. The space $BS^p(\mathbb{R}; K)$ of all Stepanov bounded functions, in conjunction with the exponent p , all measurable functions are included in this category f on \mathbb{R} in regard to value K such that $\psi^b \in L^\infty(\mathbb{R}, L^p(0, 1); K)$. $BS^p(\mathbb{R}; K)$. This is a Banach space when the norm is followed

$$\|\psi\|_{\mathcal{S}^p} = \left\| \psi^b \right\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|\psi(q)\|^p dq \right)^{1/p} < \infty. \quad (11)$$

A function $\psi \in L^p_{loc}(\mathbb{R}; K)$ is Stepanov bounded (S^p -bounded) if $\|\psi\|_{\mathcal{S}^p} < \infty$. It goes without saying that $L^p(\mathbb{R}; K) \subset BS^p(\mathbb{R}; K) \subset L^p_{loc}(\mathbb{R}; K)$.

Definition 12 (see [12]). A function $\psi \in BS^p(\mathbb{R}; K)$ is called almost periodic in the sense of Stepanov (S^p -almost periodic) if for each $\epsilon > 0$. There is such a thing as a positive number l as a result each length l interval includes a number c ; there exists $\widehat{v} \in [c, c + l_\epsilon]$ satisfying

$$\left(\int_{[t, t+1]} \|\psi(q + v\wedge) - \psi(q)\|^p dq \right)^{1/p} < \epsilon, t \in \mathbb{R}. \quad (12)$$

Let $AP^p(\mathbb{R}; K)(S^p - AP(\mathbb{R}; K))$ be the collection of all S^p -almost periodic functions.

Definition 13 (see [13]). Let $\mu \in \mathcal{N}$. A function $\psi \in BS^p(\mathbb{R}; K)$ is referred to as S^p - μ -ergodic if

$$\lim_{\omega \rightarrow +\infty} \frac{1}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \left(\int_t^{t+1} \|\psi(\mathbf{q})\|^p d\mathbf{q} \right)^{1/p} d\mu(t) = 0. \quad (13)$$

We mean by $\mathcal{E}^p(\mathbb{R}; K, \mu)$ the functions.

Definition 14 (see [13]). Let $\mu \in \mathcal{N}$. A function $\psi \in BS^p(\mathbb{R}; K)$ is called a S^p - μ -pseudo almost periodic (or Stepanov μ -pseudo almost periodic) if it can be stated in this way $\psi = \psi_1 + \psi_2$, where $\psi_1 \in AP^p(\mathbb{R}; K)$ and $\psi_2 \in \mathcal{E}^p(\mathbb{R}; K, \mu)$. $PAP^p(\mathbb{R}; K, \mu)$ or S^p - $PAP(\mathbb{R}; K, \mu)$ denotes a group of functions that are analogous.

Definition 15 (see [14]). Let $\mu \in \mathcal{N}$. A function $\psi : \mathbb{R} \times \mathbb{H} \rightarrow \mathbb{K}$ with $\psi(\cdot, y) \in L^p_{loc}(\mathbb{R}; K)$ for each $y \in \mathbb{H}$ is called S^p - μ -pseudo almost periodic in $y \in \mathbb{H}$ uniformly in $t \in \mathbb{R}$ if it may be presented as $\psi = \psi_1 + \psi_2$, where $\psi_1 \in AP^p(\mathbb{R} \times \mathbb{H}; K)$ and $\psi_2 \in \mathcal{E}^p(\mathbb{R} \times \mathbb{H}; K, \mu)$. The space occupied by such functions will be indicated by $PAPU^p(\mathbb{R} \times \mathbb{H}; K, \mu)$ or S^p - $PAP(\mathbb{R} \times \mathbb{H}; K, \mu)$.

Theorem 16 (see [14]). Let $\mu \in \mathcal{N}$ satisfy (H2), if $\psi \in PAP(\mathbb{R}; K, \mu)$ then $f \in PAP^p(\mathbb{R}; K, \mu)$.

Theorem 17 (see [18]). Assume that $\psi \in BS^p(\mathbb{R}, K)$. Then, $\psi \in \mathcal{E}^p(\mathbb{R}, K, \mu)$ if and only if, for whatever reason $\epsilon > 0$

$$\lim_{\omega \rightarrow +\infty} \frac{\mu \left\{ t \in \widehat{\omega} ; \left(\int_t^{t+1} \|\psi(\mathbf{q})\|^p d\mathbf{q} \right)^{1/p} > \epsilon \right\}}{\mu(\widehat{\omega})} = 0. \quad (14)$$

Then, by Propositions 3 and 6, as a consequence, we arrive to the following conclusion.

Proposition 18. Let $\{\psi_n\}_{n \in \mathbb{N}} \subset PAP(\mathbb{R}; K, \mu)$ be a sequence of functions. If ψ_n uniformly converges to some ψ , then $\psi \in PAP(\mathbb{R}; K, \mu)$.

3. Main Results

In the next, we are devoted whether there is unique μ -pseudo almost periodic mild solution to (1). Throughout the remainder of this paper, the following hypotheses will be required.

(H3) There exist constants $\mu_0 \geq 0$, $\vartheta \in (\pi/2, \pi)$, $L_1, L_2 \geq 0$, and $\alpha_1, \beta_1 \in (0, 1]$ with $\alpha_1 + \beta_1 > 1$ such that

$$\Sigma_\vartheta \cup \{0\} \subset \rho(A(x) - \mu_0), \|R(\lambda, A(x) - \mu_0)\| \leq \frac{L_2}{1+|\lambda|},$$

$$\begin{aligned} & \| (A(x) - \mu_0)R(\lambda, A(x) - \mu_0)[R(\mu_0, A(x)) - R(\mu_0, A(y))] \| \\ & \leq L_1 |x - y|^{\alpha_1} |\lambda|^{-\beta_1}, \end{aligned} \quad (15)$$

for $x, y \in \mathbf{R}$, $\lambda \in \Sigma_\vartheta := \{\lambda \in \mathbf{C} - \{0\} : |\arg \lambda| \leq \vartheta\}$.

(H4) The evolution family $Q(x, \widehat{\kappa})$ produced by $A(x)$ is exponentially stable, namely, there are constants $K_0, \widehat{\beta} > 0$ in order for

$$\|Q(x, \widehat{\kappa})\| \leq K_0 e^{-\widehat{\beta} \wedge (x - \widehat{\kappa})}, \quad (16)$$

for all $x \geq \widehat{\kappa}$. And the function $\mathbb{R} \times \mathbb{R} \mapsto K$, $(x, \widehat{\kappa}) \mapsto Q(x, \widehat{\kappa})$ $s \in AP(\mathbb{R} \times \mathbb{R}, K)$ uniformly for all s in any bounded subset of K .

(H5) The linear operators $\widetilde{C}_1(t), \widetilde{C}_2(t) : K \rightarrow K$ satisfy

$$t \mapsto \widetilde{C}_1(t) \text{ and } t \mapsto \widetilde{C}_2(t) \in AP(\mathbb{R}, B(K, K)). \quad (17)$$

Let

$$\omega := \max \left(\sup_{t \in \mathbb{R}} \left\| \widetilde{C}_1(t) \right\|_{B(K, K)}, \sup_{t \in \mathbb{R}} \left\| \widetilde{C}_2(t) \right\|_{B(K, K)} \right). \quad (18)$$

(H6) The function $\psi \in PAPU(\mathbb{R} \times \mathbb{H}; \mathbb{H}, \mu)$ and there exists $L_\psi > 0$ which satisfies

$$\|\psi(t, s_1) - \psi(t, s_2)\| \leq L_\psi \|s_1 - s_2\|, \quad (19)$$

for all $t \in \mathbb{R}$ and for each $s_1, s_2 \in \mathbb{H}$.

(H7) The function $g \in PAPU^p(\mathbb{R} \times K \times K; K, \mu)$ and there exists $L_3 > 0$ that satisfies $\|g(t, m_1, n_1) - g(t, m_2, n_2)\| \leq L_3 (\|m_1 - m_2\| + \|n_1 - n_2\|)$, for all $t \in \mathbb{R}$ and for $(m_1, n_1), (m_2, n_2) \in K \times K$.

Definition 19. A continuous function $z : \mathbb{R} \rightarrow K$ is referred to as a μ -pseudo almost periodic mild solution of (1), if for each $t \geq t_0$ and $t_0 \in \mathbb{R}$,

$$\begin{aligned} z(t) - \psi \left(t, \widetilde{C}_1(t)z(t) \right) &= Q(t, t_0) \left[z(t_0) - \psi \left(t_0, \widetilde{C}_1(t_0)z(t_0) \right) \right] \\ &+ \int_{t_0}^t Q(t, \mathbf{q}) g \left(\mathbf{q}, z(\mathbf{q}), \widetilde{C}_2(\mathbf{q})z(\mathbf{q}) \right) d\mathbf{q}, \quad t \in \mathbb{R}. \end{aligned} \quad (20)$$

It can be shown that a function z is a mild solution of (1) for each $t \in \mathbb{R}$ and by the expression

$$z(t) = \psi \left(t, \widetilde{C}_1(t)z(t) \right) + \int_{-\infty}^t Q(t, \mathbf{q}) g \left(\mathbf{q}, z(\mathbf{q}), \widetilde{C}_2(\mathbf{q})z(\mathbf{q}) \right) d\mathbf{q}, \quad (21)$$

since

$$\begin{aligned}
 z(t_0) &= \psi\left(t_0, \widetilde{C}_1(t_0)z(t_0)\right) + \int_{-\infty}^{t_0} Q(t_0, \mathbf{q})g\left(\mathbf{q}, z(\mathbf{q}), \widetilde{C}_2(\mathbf{q})z(\mathbf{q})\right)d\mathbf{q}, \\
 Q(t, t_0)z(t_0) &= Q(t, t_0)\psi\left(t_0, \widetilde{C}_1(t_0)z(t_0)\right) + \int_{-\infty}^{t_0} Q(t, \mathbf{q})g\left(\mathbf{q}, z(\mathbf{q}), \widetilde{C}_2(\mathbf{q})z(\mathbf{q})\right)d\mathbf{q} \\
 &= Q(t, t_0)\psi\left(t_0, \widetilde{C}_1(t_0)z(t_0)\right) + \int_{-\infty}^t Q(t, \mathbf{q})g\left(\mathbf{q}, z(\mathbf{q}), \widetilde{C}_2(\mathbf{q})z(\mathbf{q})\right)d\mathbf{q} \\
 &\quad - \int_{t_0}^t Q(t, \mathbf{q})g\left(\mathbf{q}, z(\mathbf{q}), \widetilde{C}_2(\mathbf{q})z(\mathbf{q})\right)d\mathbf{q} \\
 &= Q(t, t_0)\psi\left(t_0, \widetilde{C}_1(t_0)z(t_0)\right) + z(t) - \psi\left(t, \widetilde{C}_1(t)z(t)\right) \\
 &\quad - \int_{t_0}^t Q(t, \mathbf{q})g\left(\mathbf{q}, z(\mathbf{q}), \widetilde{C}_2(\mathbf{q})z(\mathbf{q})\right)d\mathbf{q}.
 \end{aligned} \tag{22}$$

It follows that

$$z(t) = \psi\left(t, \widetilde{C}_1(t)z(t)\right) + \int_{-\infty}^t Q(t, \mathbf{q})g\left(\mathbf{q}, z(\mathbf{q}), \widetilde{C}_2(\mathbf{q})z(\mathbf{q})\right)d\mathbf{q}. \tag{23}$$

Theorem 20. Let $\mu \in \mathcal{N}$. If $g \in PAPUP(\mathbb{R} \times W \times K; K, \mu)$ satisfies the following condition

$$\|g(t, m_1, r_1) - g(t, m_2, r_2)\|_X \leq L_3(\|m_1 - m_2\|_W + \|r_1 - r_2\|_X), \tag{24}$$

for all $t \in \mathbb{R}$ and $m_1, m_2 \in W$ and $r_1, r_2 \in K$. If $\varphi \in PAP(\mathbb{R}; K, \mu)$ moreover $\phi \in PAP(\mathbb{R}; W, \mu)$, then $g(\cdot, \phi(\cdot), \varphi(\cdot)) \in PAP^p(\mathbb{R}; K, \mu)$.

Proof. Since $g \in PAPUP(\mathbb{R} \times K, K, \mu)$, then $g = \zeta + h$ and $\phi = u + v, \varphi = x + y$, where $\zeta \in APUP(\mathbb{R} \times K \times K, K)$, $h \in \mathcal{E}^p(\mathbb{R} \times K \times K, K, \mu)$, $u, x \in AP(\mathbb{R}, K)$ and $v, y \in \mathcal{E}(\mathbb{R}, K, \mu)$. Decompose g as follows:

$$\begin{aligned}
 \Psi(t) &= \zeta(t, u(t), x(t)) + g(t, \phi(t), \varphi(t)) - \zeta(t, u(t), x(t)) \\
 &= \zeta(t, u(t), x(t)) + g(t, \phi(t), \varphi(t)) - g(t, u(t), x(t)) \\
 &\quad + h(t, u(t), x(t)).
 \end{aligned} \tag{25}$$

Let us rewrite $\Psi(t) = F(t) + \Phi(t) + \Delta(t)$, where

$$\begin{aligned}
 F(t) &= \zeta(t, u(t), x(t)), \\
 \Phi(t) &= g(t, \phi(t), \varphi(t)) - g(t, u(t), x(t)), \\
 \Delta(t) &= h(t, u(t), x(t)).
 \end{aligned} \tag{26}$$

We have $F(t) \in AP^p(\mathbb{R}, X)$ through the composition of S^p -almost periodic functions in [19]. So it can be proved $\Phi(t)$ and $\Delta(t)$ belong to $\mathcal{E}^p(\mathbb{R} \times K, K, \mu)$. Obviously, $\Phi(t) \in BS^p(\mathbb{R}, X)$. Actually, we get

$$\begin{aligned}
 &\left(\int_{[t, t+1]} \|\Phi(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} \\
 &\leq \left(\int_{[t, t+1]} (L_3(\|\phi(\mathbf{q}) - u(\mathbf{q})\| + \|\varphi(\mathbf{q}) - x(\mathbf{q})\|))^p d\mathbf{q}\right)^{1/p} \\
 &\leq L_3 \left(\left(\int_{[t, t+1]} \|v(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} + \left(\int_{[t, t+1]} \|y(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} \right), \text{ for any } t \in \mathbb{R}.
 \end{aligned} \tag{27}$$

Since v, y is bounded, then $\Phi(t) \in BS^p(\mathbb{R} \times X, X)$. Let $A_{\widehat{\omega}}^\epsilon$ and $M_{\widehat{\omega}, \epsilon}^\wedge$ be denoted by

$$\begin{aligned}
 A_{\widehat{\omega}}^\epsilon &= \left\{ t \in \widehat{\omega} ; \left(\int_{[t, t+1]} \|\Phi(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} > \epsilon \right\}, \\
 M_{\widehat{\omega}, \epsilon}^\wedge &= \left\{ t \in \widehat{\omega} ; \left(\int_{[t, t+1]} \|v(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} + \left(\int_{[t, t+1]} \|y(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} > 2\epsilon \right\}.
 \end{aligned} \tag{28}$$

From (27), we obtain $M_{\widehat{\omega}, \beta}^\wedge \subset A_{\widehat{\omega}}^\epsilon$ and $\mu(M_{\widehat{\omega}, \beta}^\wedge) \leq \mu(A_{\widehat{\omega}}^\epsilon)$ infers that

$$\begin{aligned}
 &\frac{\mu\left(\left\{t \in \widehat{\omega} ; \left(\int_{[t, t+1]} \|\Phi(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} > \epsilon\right\}\right)}{\mu(\widehat{\omega})} \\
 &\leq \frac{\mu\left(\left\{t \in \widehat{\omega} ; \left(\int_{[t, t+1]} \|v(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} > \epsilon\right\}\right)}{\mu(\widehat{\omega})} \\
 &\quad + \frac{\mu\left(\left\{t \in \widehat{\omega} ; \left(\int_{[t, t+1]} \|y(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} > \epsilon\right\}\right)}{\mu(\widehat{\omega})}.
 \end{aligned} \tag{29}$$

From Theorem 16, we get $v, y \in \mathcal{E}^p(\mathbb{R}, K, \mu)$. Since $v, y \in \mathcal{E}^p(\mathbb{R}, K, \mu)$ and Theorem 17, we derive that

$$\begin{aligned}
 &\lim_{\omega \rightarrow +\infty} \frac{\mu\left(\left\{t \in \widehat{\omega} ; \left(\int_{[t, t+1]} \|v(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} > \epsilon\right\}\right)}{\mu(\widehat{\omega})} \\
 &= \lim_{\omega \rightarrow +\infty} \frac{\mu\left(\left\{t \in \widehat{\omega} ; \left(\int_{[t, t+1]} \|y(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} > \epsilon\right\}\right)}{\mu(\widehat{\omega})} = 0,
 \end{aligned} \tag{30}$$

which deduces that

$$\lim_{\omega \rightarrow +\infty} \frac{\mu\left(\left\{t \in \widehat{\omega} ; \left(\int_{[t, t+1]} \|\Phi(\mathbf{q})\|^p d\mathbf{q}\right)^{1/p} > \epsilon\right\}\right)}{\mu(\widehat{\omega})} = 0. \tag{31}$$

This deduces from Theorem 17 that $\Phi(t) \in \mathcal{E}^p(\mathbb{R} \times K, K, \mu)$. Now, we show that $\Delta(t) \in \mathcal{E}^p(\mathbb{R}, K, \mu)$. From $u, x \in$

$AP(\mathbb{R}, K)$ then $\overline{u(\mathbb{R})}, \overline{x(\mathbb{R})}$ is compact; moreover, ζ^b is uniformly continuous with respect to $\mathbb{R} \times [0, 1] \times \overline{u(\mathbb{R})}$ (or $\overline{x(\mathbb{R})}$). We infer that h^b is uniformly continuous with respect to $\mathbb{R} \times [0, 1] \times \overline{u(\mathbb{R})}$ (or $\overline{x(\mathbb{R})}$), namely, for $\epsilon > 0$, let $\widehat{\beta} > 0$ such that for $(a_1, b_1), (a_2, b_2) \in \overline{u(\mathbb{R})} \times \overline{x(\mathbb{R})}$ with $\|a_1 - a_2\| + \|b_1 - b_2\| < \widehat{\beta}$ we get

$$\left(\int_{[t, t+1]} \|h(\mathbf{Q}, a_1, b_1) - h(\mathbf{Q}, a_2, b_2)\|^p d\mathbf{Q} \right)^{1/p} < \frac{\epsilon}{2}, \quad \text{for any } t \in \mathbb{R}. \quad (32)$$

Because $\overline{u(\mathbb{R})} \times \overline{x(\mathbb{R})}$ is compact, there exist finite balls O_θ with center $(\beta_\theta, \gamma_\theta) \in u(\mathbb{R}) \times x(\mathbb{R})$, $\theta = 1, 2, \dots, n$ such that for any $(a_2, b_2) \in \overline{u(\mathbb{R})} \times \overline{x(\mathbb{R})}$ we get $\|\beta_\theta - a_2\| + \|\gamma_\theta - b_2\| < \widehat{\beta}$ for $1 \leq \theta \leq n$. Let

$$\begin{aligned} \mathfrak{B}_\theta &= \left\{ t \in \mathbb{R}, \|u(t) - \beta_\theta\| + \|x(t) - \gamma_\theta\| < \widehat{\beta} \right\}, \quad \theta = 1, 2, 3, \dots, n, \\ \mathfrak{D}_1 &= \mathfrak{B}_1, \\ \mathfrak{D}_\theta &= \frac{\mathfrak{B}_\theta}{\left(\bigcup_{i=1}^{\theta-1} \mathfrak{B}_i \right)}, \quad \theta = 1, 2, 3, \dots, n. \end{aligned} \quad (33)$$

Thus,

$$\begin{aligned} \mathbb{R} &= \bigcup_{\theta=1}^n \mathfrak{B}_\theta, \\ \mathbb{R} &= \bigcup_{\theta=1}^n \mathfrak{D}_\theta, \\ \mathfrak{D}_i \cap \mathfrak{D}_j &= \emptyset (i \neq j). \end{aligned} \quad (34)$$

Define function $e_1, e_2 : \mathbb{R} \rightarrow K$ by $e_1(t) = \beta_\theta, e_2(t) = \gamma_\theta$ for $t \in B_\theta$ and $\theta = 1, 2, 3, \dots, n$. So $\|u(t) - e_1(t)\| + \|x(t) - e_2(t)\| < \widehat{\beta}$ for $t \in \mathbb{R}$. Therefore, we have

$$\left(\int_{[t, t+1]} \|h(\mathbf{Q}, u(\mathbf{Q}), x(\mathbf{Q})) - h(\mathbf{Q}, e_1(\mathbf{Q}), e_2(\mathbf{Q}))\|^p d\mathbf{Q} \right)^{1/p} \leq 4 \times 2^{1/p} \epsilon. \quad (35)$$

Since $h \in \mathcal{E}^p(\mathbb{R} \times K, K, \mu)$, then there exists $\omega_0 > 0$ such that for $\widehat{\omega} > \omega_0$: for every $\theta = 1, 2, 3, \dots, n$

$$\frac{1}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \left(\int_{[t, t+1] \cap \mathfrak{D}_\theta} \|h(\mathbf{Q}, \beta_\theta, \gamma_\theta)\|^p d\mathbf{Q} \right)^{1/p} d\mu(t) \leq 4 \times 2^{1/p} n^{1/p} \epsilon. \quad (36)$$

Then,

$$\begin{aligned} \frac{1}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \left(\int_{[t, t+1]} \|h(\mathbf{Q}, u(\mathbf{Q}), x(\mathbf{Q}))\|^p d\mathbf{Q} \right)^{1/p} d\mu(t) &= \frac{1}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \\ &\cdot \left(\sum_{\theta=1}^n \int_{[t, t+1] \cap \mathfrak{D}_\theta} \|h(\mathbf{Q}, u(\mathbf{Q}), x(\mathbf{Q})) - h(\mathbf{Q}, \beta_\theta, \gamma_\theta) + h(\mathbf{Q}, \beta_\theta, \gamma_\theta)\|^p d\mathbf{Q} \right)^{1/p} d\mu(t) \\ &\leq \frac{2^{1+(1/p)}}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \left(\sum_{\theta=1}^n \int_{[t, t+1] \cap \mathfrak{D}_\theta} \|h(\mathbf{Q}, u(\mathbf{Q}), x(\mathbf{Q})) - h(\mathbf{Q}, \beta_\theta, \gamma_\theta)\|^p d\mathbf{Q} \right)^{\frac{1}{p}} d\mu(t) \\ &\quad + \frac{2^{1+(1/p)}}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \left(\sum_{\theta=1}^n \int_{[t, t+1] \cap \mathfrak{D}_\theta} \|h(\mathbf{Q}, \beta_\theta, \gamma_\theta)\|^p d\mathbf{Q} \right)^{1/p} d\mu(t) \\ &\leq \frac{2^{1+(1/p)}}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \left(\int_{[t, t+1]} \|h(\mathbf{Q}, u(\mathbf{Q}), x(\mathbf{Q})) - h(\mathbf{Q}, e_1(\mathbf{Q}), e_2(\mathbf{Q}))\|^p d\mathbf{Q} \right)^{1/p} d\mu(t) \\ &\quad + \frac{2^{1+(1/p)}}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \left(\sum_{\theta=1}^n \int_{[t, t+1] \cap \mathfrak{D}_\theta} \|h(\mathbf{Q}, \beta_\theta, \gamma_\theta)\|^p d\mathbf{Q} \right)^{1/p} d\mu(t) \leq \epsilon. \end{aligned} \quad (37)$$

It follows that $\Delta(t) \in \mathcal{E}^p(\mathbb{R}, K, \mu)$. The proof is completed. \square

Lemma 21. Let $\mu \in \mathcal{N}$. Assume that (H3) – (H4) hold. If Φ is an S^p - μ -pseudo almost periodic function, then

$$(\Gamma \Phi)(t) = \int_{-\infty}^t Q(t, \mathbf{Q}) \Phi(\mathbf{Q}) d\mathbf{Q}, \quad t \in \mathbb{R} \quad (38)$$

is a μ -pseudo almost periodic function.

Proof. Since $\Phi \in S^p P A P(\mathbb{R}; K, \mu)$, then we have the following decomposition $\Phi = \Phi_1 + \Phi_2$ where $\Phi_1 \in S^p A P(\mathbb{R}; K)$ and $\Phi_2 \in \mathcal{E}^p(\mathbb{R}; K, \mu)$. We define

$$\begin{aligned} (\Gamma \Phi_1)(t) &= \int_{-\infty}^t Q(t, \mathbf{Q}) \Phi_1(\mathbf{Q}) d\mathbf{Q}, \quad (\Gamma \Phi_2)(t) \\ &= \int_{-\infty}^t Q(t, \mathbf{Q}) \Phi_2(\mathbf{Q}) d\mathbf{Q}, \quad t \in \mathbb{R}. \end{aligned} \quad (39)$$

Since there exist $(1/p) + (1/q) = 1$, where $p > 1$, the proof is as follows. Let us show that $(\Gamma \Phi_1)(t) \in AP(\mathbb{R}; K)$.

We define

$$(\Gamma \Phi_1)_\theta(t) = \int_{t-\theta}^{t-\theta+1} Q(t, \mathbf{Q}) \Phi_1(\mathbf{Q}) d\mathbf{Q}, \quad \theta \in \mathbb{N}, \quad t \in \mathbb{R}. \quad (40)$$

By virtue of the Hölder inequality and formula (16), we get

$$\begin{aligned} \|(\Gamma \Phi_1)_\theta(t)\| &= \left\| \int_{t-\theta}^{t-\theta+1} Q(t, \mathbf{Q}) \Phi_1(\mathbf{Q}) d\mathbf{Q} \right\| \\ &\leq \int_{t-\theta}^{t-\theta+1} \|Q(t, \mathbf{Q})\| \|\Phi_1(\mathbf{Q})\| d\mathbf{Q} \\ &\leq K_0 \left[\int_{t-\theta}^{t-\theta+1} e^{-q\beta^\wedge(t-\mathbf{Q})} d\mathbf{Q} \right]^{1/q} \left[\int_{t-\theta}^{t-\theta+1} \|\Phi_1(\mathbf{Q})\|^p d\mathbf{Q} \right]^{1/p} \\ &\leq \frac{K_0 (e^{q\beta^\wedge} - 1)}{\sqrt[q]{q\widehat{\beta}}} e^{-\theta\beta^\wedge} \|\Phi_1\|_{S^p}. \end{aligned} \quad (41)$$

Since $\sum_{\theta=1}^{\infty} e^{-\beta\wedge\theta}$ is convergent, so take advantage of the Weierstrass theorem, we obtain that the sequence of functions $\sum_{\theta=1}^{\infty} (\Gamma\Phi_1)_{\theta}(t)$ is convergent uniformly on \mathbb{R} ; moreover, $\sum_{\theta=1}^{\infty} (\Gamma\Phi_1)_{\theta}(t) = (\Gamma\Phi_1)(t)$ then $(\Gamma\Phi_1)(\cdot)$ is continuous and

$$\begin{aligned} \|(\Gamma\Phi_1)(t)\| &\leq \sum_{\theta=1}^{\infty} \|(\Gamma\Phi_1)_{\theta}(t)\| \\ &\leq \frac{K_0(e^{q\beta\wedge} - 1)}{\sqrt[q]{q\widehat{\beta}}} \|\Phi_1\|_{\mathcal{S}^p} \sum_{\theta=1}^{\infty} e^{-\beta\wedge\theta}, \quad \text{for all } t \in \mathbb{R}. \end{aligned} \quad (42)$$

Next, we can demonstrate that $(\Gamma\Phi_1)(\cdot) \in AP(\mathbb{R}; K)$. Since $\Phi_1 \in \mathcal{S}^p AP(\mathbb{R}; K)$, let $\epsilon > 0$; afterwards, there is $l(\epsilon) > 0$ such that each length interval $l(\epsilon)$ contains \widehat{v} based on the estimate

$$\sup_{t \in \mathbb{R}} \left[\int_t^{t+\widehat{v}} \|\Phi_1(\mathfrak{q} + v\wedge) - \Phi_1(\mathfrak{q})\|^p d\mathfrak{q} \right]^{1/p} < \epsilon_1 \epsilon, \quad (43)$$

where $\epsilon_1 = \sqrt[q]{q\widehat{\beta}}(e^{\beta\wedge} - 1)/(K_0(e^{q\beta\wedge} - 1))$,

$$\begin{aligned} &\|(\Gamma\Phi_1)_{\theta}(t + \widehat{v}) - (\Gamma\Phi_1)_{\theta}(t)\| \\ &= \left\| \int_{t+\widehat{v}-\theta}^{t+\widehat{v}-\theta+1} Q(t + \widehat{v}, \mathfrak{q})\Phi_1(\mathfrak{q})d\mathfrak{q} - \int_{t-\theta}^{t-\theta+1} Q(t, \mathfrak{q})\Phi_1(\mathfrak{q})d\mathfrak{q} \right\| \\ &\leq \int_{t-\theta}^{t-\theta+1} \|Q(t, \mathfrak{q})\| \|\Phi_1(\mathfrak{q} + \widehat{v}) - \Phi_1(\mathfrak{q})\| d\mathfrak{q} \\ &\leq K_0 \left[\int_{t-\theta}^{t-\theta+1} e^{-q\beta\wedge(t-\mathfrak{q})} d\mathfrak{q} \right]^{1/q} \left[\int_{t-\theta}^{t-\theta+1} \|\Phi_1(\mathfrak{q} + v\wedge) - \Phi_1(\mathfrak{q})\|^p d\mathfrak{q} \right]^{1/p} \\ &\leq \frac{K_0(e^{q\beta\wedge} - 1)}{\sqrt[q]{q\widehat{\beta}}} e^{-\theta\beta\wedge} \epsilon_1 \epsilon. \end{aligned} \quad (44)$$

Hence,

$$\sum_{\theta=1}^{\infty} \|(\Gamma\Phi_1)_{\theta}(t + \widehat{v}) - (\Gamma\Phi_1)_{\theta}(t)\| < \epsilon \epsilon_1 \frac{K_0(e^{q\beta\wedge} - 1)}{\sqrt[q]{q\widehat{\beta}}} \sum_{\theta=1}^{\infty} e^{-\theta\beta\wedge} = \epsilon. \quad (45)$$

Therefore, $\sum_{\theta=1}^{\infty} (\Gamma\Phi_1)_{\theta}(\cdot) \in AP(\mathbb{R}, K)$ for all $\theta \in \mathbb{N}$ and $(\Gamma\Phi_1)(\cdot) \in AP(\mathbb{R}; K)$.

Next, we show that $(\Gamma\Phi_2)(\cdot) \in \mathcal{Z}(\mathbb{R}; K, \mu)$.

Let $\omega > 0$, $\Phi_2 \in \mathcal{Z}^p(\mathbb{R}; K, \mu)$; we get that

$$\lim_{\omega \rightarrow +\infty} \frac{1}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \left(\int_{\mathfrak{Q}} \|\Phi_2(\sigma)\|^p d\sigma \right)^{1/p} d\mu(\mathfrak{Q}) = 0. \quad (46)$$

First, we prove that $\Gamma\Phi_2 \in BC(\mathbb{R}; K)$. As a matter of fact, it is similar to above work of $\Gamma\Phi_1$. Then, we demonstrate that $\Gamma\Phi_2 \in \mathcal{Z}(\mathbb{R}; K, \mu)$. We observe

$$(\Gamma\Phi_2)_{\theta}(t) = \int_{t-\theta}^{t-\theta+1} Q(t, \mathfrak{q})\Phi_2(\mathfrak{q})d\mathfrak{q}, \theta \in \mathbb{N}, \quad t \in \mathbb{R}. \quad (47)$$

Taking advantage of the Hölder inequality and applying formula (16), considering that

$$\begin{aligned} \|(\Gamma\Phi_2)_{\theta}(t)\| &= \left\| \int_{t-\theta}^{t-\theta+1} Q(t, \mathfrak{q})\Phi_2(\mathfrak{q})d\mathfrak{q} \right\| \\ &\leq \int_{t-\theta}^{t-\theta+1} \|Q(t, \mathfrak{q})\| \|\Phi_2(\mathfrak{q})\| d\mathfrak{q} \\ &\leq K_0 \left[\int_{t-\theta}^{t-\theta+1} e^{-q\beta\wedge(t-\mathfrak{q})} d\mathfrak{q} \right]^{1/q} \left[\int_{t-\theta}^{t-\theta+1} \|\Phi_2(\mathfrak{q})\|^p d\mathfrak{q} \right]^{1/p} \\ &\leq \frac{K_0(e^{q\beta\wedge} - 1)}{\sqrt[q]{q\widehat{\beta}}} e^{-\theta\beta\wedge} \|\Phi_2\|_{\mathcal{S}^p}, \end{aligned} \quad (48)$$

it follows that

$$\frac{1}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \|(\Gamma\Phi_2)_{\theta}(t)\| d\mu(t) \leq \frac{K_0(e^{q\beta\wedge} - 1)}{\sqrt[q]{q\widehat{\beta}}} e^{-\theta\beta\wedge} \frac{1}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \|\Phi_2(t)\|_{\mathcal{S}^p} d\mu(t),$$

$$\begin{aligned} &\lim_{\omega \rightarrow +\infty} \frac{1}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \|(\Gamma\Phi_2)_{\theta}(t)\| d\mu(t) \\ &\leq \frac{K_0(e^{q\beta\wedge} - 1)}{\sqrt[q]{q\widehat{\beta}}} e^{-\theta\beta\wedge} \cdot \lim_{\omega \rightarrow +\infty} \frac{1}{\mu(\widehat{\omega})} \int_{\widehat{\omega}} \|\Phi_2(t)\|_{\mathcal{S}^p} d\mu(t) = 0. \end{aligned} \quad (49)$$

Hence, $(\Gamma\Phi_2)_{\theta} \in \mathcal{Z}(\mathbb{R}, K, \mu)$. For another, since $\sum_{\theta=1}^{\infty} e^{-\beta\wedge\theta}$ is convergent, so applying the Weierstrass theorem, then the series, as a result, $\sum_{\theta=1}^{\infty} (\Gamma\Phi_2)_{\theta}(t)$ is convergent uniformly on \mathbb{R} .

What is more, $\sum_{\theta=1}^{\infty} (\Gamma\Phi_2)_{\theta}(t) = (\Gamma\Phi_2)(t)$; therefore, $\sum_{\theta=1}^{\infty} (\Gamma\Phi_2)_{\theta}(t) \in \mathcal{Z}(\mathbb{R}, K, \mu)$, so $(\Gamma\Phi_2)(t) \in \mathcal{Z}(\mathbb{R}, K, \mu)$ from Proposition 18. \square

Theorem 22. Let $\mu \in \mathcal{N}$ and suppose that (H3)–(H7) hold. If $L_{\psi}\omega + (2K_0L_3/\widehat{\beta})(1 + \omega) < 1$, then equation (1) has a unique μ -pseudo almost periodic mild solution.

Proof. Define the operator Π as follows:

$$(\Pi z)(t) = \psi\left(t, \widetilde{C}_1(t)z(t)\right) + \int_{-\infty}^t Q(t, \mathfrak{q})g\left(\mathfrak{q}, z(\mathfrak{q}), \widetilde{C}_2(\mathfrak{q})z(\mathfrak{q})\right)d\mathfrak{q}. \quad (50)$$

For $z(t) \in PAP(\mathbb{R}; K, \mu)$, by Lemma 21 in [5], we get that $\widetilde{C}_1(t)z(t)$ is μ -pseudo almost periodic; by (H6) and Theorem 10, it is simple to demonstrate that $\psi(t, \widetilde{C}_1(t)z(t))$ is part of $PAP(\mathbb{R}; K, \mu)$. We will demonstrate this now that $\int_{-\infty}^t Q(t, \mathfrak{q})g(\mathfrak{q}, z(\mathfrak{q}), \widetilde{C}_2(\mathfrak{q})z(\mathfrak{q}))d\mathfrak{q}$ is part of $PAP(\mathbb{R}; K, \mu)$;

by using Theorem 20 and hypothesis (H7), we can easily deduce that $g(\mathbf{q}, z(\mathbf{q}), \widetilde{C}_2(\mathbf{q})z(\mathbf{q}))$ is part of $PAP^p(\mathbb{R}; K, \mu)$. Thus, Π is mapping from $PAP(\mathbb{R}; K, \mu)$ into itself.

Let $z, v \in PAP(\mathbb{R}; K, \mu)$. We deduce that

$$\begin{aligned} \|\Pi z - \Pi v\| &\leq \left\| \psi\left(t, \widetilde{C}_1(t)z(t)\right) - \psi\left(t, \widetilde{C}_1(t)v(t)\right) \right\| + \int_{-\infty}^t \|Q(t, \mathbf{q})\| \\ &\quad \cdot \left\| g\left(\mathbf{q}, z(\mathbf{q}), \widetilde{C}_2(\mathbf{q})z(\mathbf{q})\right) - g\left(\mathbf{q}, v(\mathbf{q}), \widetilde{C}_2(\mathbf{q})v(\mathbf{q})\right) \right\| d\mathbf{q} \\ &\leq L_\psi \left\| \widetilde{C}_1(t)z(t) - \widetilde{C}_1(t)v(t) \right\| + K_0 L_3 \int_{-\infty}^t e^{-\beta\Lambda(t-\mathbf{q})} \\ &\quad \cdot \left(\left(1 + \|\widetilde{C}_2(\mathbf{q})\|\right) \|z(\mathbf{q}) - v(\mathbf{q})\| \right) d\mathbf{q} \\ &\leq L_\psi \left\| \widetilde{C}_1(t) \right\| \|z - v\|_\infty + 2K_0 L_3 \\ &\quad \cdot \left(1 + \|\widetilde{C}_2(\mathbf{q})\|\right) \left(\int_{-\infty}^t e^{-\beta\Lambda(t-\mathbf{q})} d\mathbf{q} \right) \|z - v\|_\infty \\ &\leq \left(L_\psi \omega + \frac{2K_0}{\beta} (1 + \omega) L_3 \right) \|z - v\|_\infty. \end{aligned} \quad (51)$$

Thus,

$$\|\Pi z - \Pi v\|_\infty \leq \left(L_\psi \omega + \frac{2K_0}{\beta} (1 + \omega) L_3 \right) \|z - v\|_\infty. \quad (52)$$

As a result, this establishes that Π is a contraction map. We may infer that Π has a single fixed point in $PAP(\mathbb{R}; K, \mu)$ such that $\Pi z = z$. Consequently, (1) has a unique μ -pseudo almost periodic mild solution. \square

4. Example

As an example, take a look at the differential equations listed below.

$$\begin{aligned} &\frac{d}{dz} [u(z, x) - \psi(z, c_1(z, x)u(z, x))] \\ &= \frac{d^2}{dx^2} [u(z, x) - \psi(z, c_1(z, x)u(z, x))] \\ &\quad + (-3 + \sin(mz) + \sin(nz)) [u(z, x) - \psi(z, c_1(z, x)u(z, x))] \\ &\quad + g(z, u(z, x), c_2(z, x)u(z, x)), \quad z \in \mathbb{R}, x \in [0, 1], \end{aligned} \quad (53)$$

$$\begin{aligned} u(z, 0) - \psi(z, c_1(z, 0)u(z, 0)) &= u(z, 1) - \psi(z, c_1(z, 1)u(z, 1)) \\ &= 0, \quad z \in \mathbb{R}, \end{aligned} \quad (54)$$

where $m, n \in \mathbb{R}$ and $m/n \notin \mathbb{Q}$, $\psi, g : \mathbb{R} \times K \mapsto K$, and $c_1, c_2 : \mathbb{R} \times [0, 1] \mapsto \mathbb{R}$ are continuous functions. So as to rephrase this equation in form (1.1), we set the spaces $K = L^2([0, 1])$ provided with the norm $\|\cdot\|_2$. The operator $A : D(A) \subset K \rightarrow K$ is defined by

$$Au = u'' \quad u \in D(A). \quad (55)$$

Furthermore, A is the infinitesimally small generator C_0 -semigroup $\{T(z)\}_{z \geq 0}$ with $\|T(z)\| \leq e^{-z}$ ($K = \beta = 1$) for $z \geq 0$.

We consider the operator $A(z)$ given by

$$\begin{cases} D(A(z)) = D(A), \\ A(z)v = (A - 3 + \sin(mz) + \sin(nz))v, \quad \text{for all } v \in D(A). \end{cases} \quad (56)$$

Thus, $D(A(z)) = D(A)$. Furthermore,

$$\begin{aligned} \|A(z) - A(s)\| &= \|(\sin(mz) - \sin(ms) + \sin(nz) - \sin(ns))\| \\ &\leq (|m| + |n|)|z - s|, \end{aligned} \quad (57)$$

for every $s, z \in \mathbb{R}$, and hence, (H3) holds. Then, an evolution family $Q(z, s)_{z \geq s}$ generated by $A(z)$ with

$$Q(z, s)v = T(z - s) \exp \left(\int_s^z (-3 + \sin(m\widehat{v}) + \sin(n\widehat{v})) d\widehat{v} \right) v. \quad (58)$$

Since $\|Q(z, s)\| \leq e^{-(z-s)}$, then it is very clear to know that $A(z)$ fulfill the hypothesis (H4) with $K_0 = 1, \beta = 1$. At the same time, we define

$$\begin{aligned} \widetilde{C}_1(z)u &= c_1(z, x)u, \quad \text{for all } z \in \mathbb{R}, \\ \widetilde{C}_2(z)u &= c_2(z, x)u, \quad \text{for all } z \in \mathbb{R}. \end{aligned} \quad (59)$$

$c_1(z, \cdot), c_2(z, \cdot) : \mathbb{R} \times [0, 1] \mapsto \mathbb{R}$ are continuous and almost periodic in $z \in \mathbb{R}$ uniformly in $x \in [0, 1]$. Then, Equation (53) can be represented as the abstract equation (1).

Study the measure μ with respect to its Radon-Nikodym derivative of

$$\rho(z) = \begin{cases} e^z, & \text{if } z \leq 0, \\ 1, & \text{if } z > 0. \end{cases} \quad (60)$$

Then, $\mu \in \mathcal{N}$ satisfy (H1) and (H2). If we take the assumption that ψ, g satisfy (H6) and (H7), therefore, all assumptions of Theorem 22 are fulfilled; thus, the evolution equation (53) has a unique μ -pseudo almost periodic solution if L_ψ, L_3 is sufficiently small.

5. Conclusions and the Future Work

The primary objective of this research is to find out whether there is unique μ -pseudo almost periodic solution to a functional evolution equation with Stepanov forcing components in a Banach space using measure theory, evolution family, and Hölder inequality. The Lipschitz condition and the contraction mapping theory are used to develop certain suitable conditions that guarantee whether there is unique μ -pseudo almost periodic solutions to the problem. As a future work, we intend to study whether there is unique Stepanov-like

pseudo almost periodic solution of neural network and discrete equation. We will also use the Hölder inequality to consider whether there is a unique Stepanov-like pseudo almost periodic solution of discrete equation. Almost periodic problems play a very important basic role in engineering applications and mathematical research.

Data Availability

The data used to support the findings of this study are included in the article.

Conflicts of Interest

The author states that the publishing of this paper does not include any conflicts of interest.

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