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Research Article

Chen-Ricci Inequalities with a Quarter Symmetric Connection in Generalized Space Forms

Ali H. Al-Khaldi,[1] Mohd. Aquib ,[2] Mohd Aslam,[3] and Meraj Ali Khan [4]

[1]*Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha, Saudi Arabia*
[2]*Department of Mathematics, Sri Venkateswara College, University of Delhi, New Delhi 110021, India*
[3]*Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia, New Delhi 110025, India*
[4]*Department of Mathematics, University of Tabuk, Tabuk, Saudi Arabia*

Correspondence should be addressed to Mohd. Aquib; aquib80@gmail.com

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In this article, we obtain improved Chen-Ricci inequalities for submanifolds of generalized space forms with quarter-symmetric metric connection, with the help of which we completely characterized the Lagrangian submanifold in generalized complex space form and a Legendrian submanifold in a generalized Sasakian space form. We also discuss some geometric applications of the obtained results.

1. Introduction

One of the most basic problems in submanifold theory is to develop a simple relationship between the extrinsic invariants and the intrinsic invariants. The sectional curvature, the scalar curvature, and the Ricci curvature are the main intrinsic invariants while the squared mean curvature is the main extrinsic invariant.

Chen obtained the following important bound of the Ricci curvature Ric in terms of the mean curvature $\mathscr{H}$ for Lagrangian submanifolds in complex space forms [1]:

$$\mathrm{Ric} \leq (m-1)c + \frac{m^2}{4}\|\mathscr{H}\|^2, \tag{1}$$

where c is the constant holomorphic sectional curvature of the complex space form.

Further, he discussed the geometry of a Lagrangian submanifold satisfying the equality case of the inequality under the condition that the dimension of the kernel of the second fundamental form is constant. The inequality (1) is known as the Chen-Ricci inequality. This inequality attracted many researchers due to its geometric importance [2–12].

Deng [13] improved the above inequality as

$$\mathrm{Ric}(U) \leq \frac{m-1}{4}\left(c + m\|\mathscr{H}\|^2\right). \tag{2}$$

In [14], Deng further extended his result for Lagrangian submanifolds in quaternion space forms. In [15], Tripathi improved the inequality in the case of curvature-like tensors. In [6], Mihai and Radulescu obtained the same relation in Sasakian space forms using semisymmetric connection as

$$\mathrm{Ric}(U) + (m-2)\alpha(U,U) + tr\alpha \leq \frac{m-1}{4}\left(c + 3 + m\|\mathscr{H}\|^2\right). \tag{3}$$

As the curvature invariants are of great interest in theoretical physics (see [16]), the above studies motivate us to obtain a complete characterization of Lagrangian submanifold in generalized complex space form and a Legendrian submanifold in a generalized Sasakian space form.

2. Preliminaries

Let N be a Riemannian manifold and $\bar{\nabla}$ be a linear connection on N. Then, $\bar{\nabla}$ is said to be a semisymmetric connection if its torsion tensor T satisfies

$$T(U, V) = \pi(V)U - \pi(U)V, \tag{4}$$

for a 1-form π, then the connection $\bar{\nabla}$ is called a semisymmetric connection [17]. Let g be a Riemannian metric on N. If $\bar{\nabla} g = 0$, then $\bar{\nabla}$ is called a semisymmetric metric connection on N. The semisymmetric metric connection $\bar{\nabla}$ on N is given by

$$\bar{\nabla}_U V = \bar{\nabla}_U V + \pi(V)U - g(U, V)\Gamma, \tag{5}$$

for any U, V on N, where $\bar{\nabla}$ denotes the Levi-Civita connection with respect to Riemannian metric g and Γ is a vector field. Further, $\bar{\nabla}$ is said to be a semisymmetric nonmetric connection if it satisfies

$$\bar{\nabla}_U V = \bar{\nabla}_U V + \pi(V)U. \tag{6}$$

Moreover, the linear connection $\bar{\nabla}$ on a Riemannian manifold N with Riemannian metric g is said to be a quarter-symmetric connection if its torsion tensor T is given by

$$T(U, V) = \bar{\nabla}_U V - \bar{\nabla}_V U - [U, V], \tag{7}$$

which satisfies

$$T(U, V) = \pi(V)\phi U - \pi(U)\phi V, \tag{8}$$

such that π is a 1-form given by

$$\pi(U) = g(U, \Gamma), \tag{9}$$

where Γ is a vector field and ϕ is a (1,1) tensor field.

Then, we can define a special quarter-symmetric connection by

$$\bar{\nabla}_U V = \bar{\nabla}_U V + \psi_1 \pi(V)U - \psi_2 g(U, V)\Gamma, \tag{10}$$

where ψ_1 and ψ_2 are real constants.

Remark 1. We notice from (5) that [18]

(1) if $\psi_1 = \psi_2 = 1$, then a quarter symmetric connection becomes a semisymmetric metric connection

(2) if $\psi_1 = 1$ and $\psi_2 = 0$, then a quarter-symmetric connection becomes a semisymmetric nonmetric connection

Remark 2. It is also worthy to mention here that the quarter symmetric connections generalized several well-known connections.

The curvature tensor $\bar{R}$ with respect to $\bar{\nabla}$ is

$$\bar{R}(U, V)Z = \bar{\nabla}_U \bar{\nabla}_V Z - \bar{\nabla}_V \bar{\nabla}_U Z - \bar{\nabla}_{[U,V]} Z. \tag{11}$$

In the same way, we can also define the curvature tensor $\tilde{\bar{R}}$.

Let

$$\begin{aligned} \beta_1(U, V) &= \left(\bar{\nabla}_U \pi\right)(V) - \psi_1 \pi(U)\pi(V) + \frac{\psi_2}{2} g(U, V)\pi(\Gamma), \\ \beta_2(U, V) &= \frac{\pi(\Gamma)}{2} g(U, V) + \pi(U)\pi(V), \end{aligned} \tag{12}$$

are $(0, 2)$ tensors. Then, the curvature tensor of N is given by [19]

$$\begin{aligned} \bar{R}(U, V, Z, W) = \tilde{\bar{R}}(U, V, Z, W) &+ \psi_1 \beta_1(U, Z) g(V, W) \\ &- \psi_1 \beta_1(V, Z) g(U, W) + \psi_2 \beta_1(V, W) g(U, Z) \\ &- \psi_2 \beta_1(U, W) g(V, Z) + \psi_2(\psi_1 - \psi_2) g(U, Z)\beta_2(V, W) \\ &- \psi_2(\psi_1 - \psi_2) g(V, Z)\beta_2(U, W). \end{aligned} \tag{13}$$

Let $\mathscr{M}$ be an m-dimensional submanifold in a Riemannian manifold N. Let ∇ and $\tilde{\nabla}$ be the induced quarter symmetric-metric connection and Levi-Civita connection, respectively, on $\mathscr{M}$. Then, the Gauss formulas are

$$\begin{aligned} \bar{\nabla}_U V &= \nabla_U V + \zeta(U, V), \quad U, V \in \Gamma(T\mathscr{M}), \\ \bar{\nabla}_U V &= \tilde{\nabla}_U V + \tilde{\zeta}(U, V), \quad U, V \in \Gamma(T\mathscr{M}), \end{aligned} \tag{14}$$

where $\tilde{\zeta}$ is the second fundamental form that satisfies the relation

$$\zeta(U, V) = \tilde{\zeta}(U, V) - \psi_2 g(U, V)\Gamma^{\perp}, \tag{15}$$

where $\Gamma^{\perp}$ is the normal component of the vector field Γ on $\mathscr{M}$.

Moreover, the equation of Gauss is defined by [19]

$$\begin{aligned} \bar{R}(U, V, Z, W) = R(U, V, Z, W) &- g(\zeta(U, W), \zeta(V, Z)) \\ &+ g(\zeta(V, W), \zeta(U, Z)) \\ &+ (\psi_1 - \psi_2) g\left(\zeta(V, Z), \Gamma^{\perp}\right) g(U, W) \\ &+ (\psi_2 - \psi_1) g\left(\zeta(U, Z), \Gamma^{\perp}\right) g(V, W). \end{aligned} \tag{16}$$

3. Characterization of Lagrangian Submanifold in Generalized Complex Space Form

A smooth manifold N endowed with an almost complex structure J and a Riemannian metric g that is compatible with J is called an almost Hermitian manifold. Further, for the Levi-Civita connection ∇ if $\nabla J = 0$, then an almost Hermitian manifold is said to be a Kaehler manifold. A Kaehler manifold of constant holomorphic curvature is called a complex space form. The curvature tensor of a complex space form is given by

$$\begin{aligned}\tilde{\tilde{R}}(U, V, Z, W) = \frac{c}{4}\{&g(V, Z)g(U, W) - g(U, Z)g(V, W)\\ &+ g(U, JZ)g(JV, W) - g(V, JZ)g(JU, W)\\ &+ 2g(U, JV)g(JZ, W)\}.\end{aligned} \tag{17}$$

However, an almost Hermitian manifold N is called a generalized complex space form [20–22], denoted by $N(f_1, f_2)$, if for all vector fields U, V, and Z on N, the Riemannian curvature tensor $\tilde{\tilde{R}}$ satisfies

$$\begin{aligned}\tilde{\tilde{R}}(U, V, Z, W) = &f_1\{g(V, Z)g(U, W) - g(U, Z)g(V, W)\}\\ &+ f_2\{g(U, JZ)g(JV, W) - g(V, JZ)g(JU, W)\\ &+ 2g(U, JV)g(JZ, W)\},\end{aligned} \tag{18}$$

where f_1 and f_2 are smooth functions on N.

In fact, we have following fundamental result from Tricerri and Vanhecke [20].

Theorem 3 (see [20]). *Let N be a connected almost Hermitian manifold with real dimension $2m > 6$ and Riemannian curvature $\tilde{\tilde{R}}$ is of the form (18) such that f_2 is not identically zero. Then, N is a complex space form.*

Remark 4. From (18), we notice that if $f_1 = f_2 = c/4$, then we recover the complex space form.

From (13) and (18), we have

$$\begin{aligned}\bar{R}(U, V, Z, W) = &f_1\{g(V, Z)g(U, W) - g(U, Z)g(V, W)\}\\ &+ f_2\{g(U, JZ)g(JV, W) - g(V, JZ)g(JU, W)\\ &+ 2g(U, JV)g(JZ, W)\} + \psi_1\beta_1(U, Z)g(V, W)\\ &- \psi_1\beta_1(V, Z)g(U, W) + \psi_2\beta_1(V, W)g(U, Z)\\ &- \psi_2\beta_1(U, W)g(V, Z) + \psi_2(\psi_1 - \psi_2)g(U, Z)\beta_2(V, W)\\ &- \psi_2(\psi_1 - \psi_2)g(V, Z)\beta_2(U, W).\end{aligned} \tag{19}$$

Lemma 5 (see [13]). *Let $f_1(u_1, u_2, \cdots, u_m)$ be a function on $\mathbb{R}^m$ defined by*

$$f_1(u_1, u_2, \cdots, u_m) = u_1 \sum_{j=2}^{m} u_j - \sum_{j=2}^{m} u_j^2. \tag{20}$$

If $u_1 + u_2 + \cdots + u_m = 2ma$, then

$$f_1(u_1, u_2, \cdots, u_m) \le \frac{m-1}{4m}(u_1 + u_2 + \cdots + u_m)^2, \tag{21}$$

and the equality holds if and only if $(1/(m+1))u_1 = u_2 = \cdots = u_m = a$, where a is a constant.

Lemma 6 (see [13]). *Let $f_2(u_1, u_2, \cdots, u_m)$ be a function on $\mathbb{R}^m$ defined by*

$$f_2(u_1, u_2, \cdots, u_m) = u_1 \sum_{j=2}^{m} u_j - u_1^2. \tag{22}$$

If $u_1 + u_2 + \cdots + u_m = 4a$, then

$$f_2(u_1, u_2, \cdots, u_m) \le \frac{1}{8}(u_1 + u_2 + \cdots + u_m)^2, \tag{23}$$

and the equality holds if and only if $u_1 = a$ and $u_2 + \cdots + u_m = 3a$, where a is a constant.

Let M^m be an m-dimensional submanifold of an almost Hermitian manifold N. Then, M^m is said to be totally real if

$$J\left(T_pM^m\right) \subset T_p^{\perp}M^m. \tag{24}$$

Then, we have the following relations [23]:

$$\tilde{A}_{JU}V = \tilde{A}_{JV}U, \quad U, V \in T_pM, \tag{25}$$

or equivalently,

$$\tilde{\zeta}_{ij}^{k} = \tilde{A}_{ik}^{j} = \tilde{\zeta}_{jk}^{\ i}, \quad \forall i, j, k = 1, \cdots m, \tag{26}$$

where $\tilde{A}^k$ is the shape operator with respect to $\bar{\nabla}$ and

$$\tilde{\zeta}_{ij}^{\ k} = g\left(\tilde{\zeta}\left(e_i, e_j\right), Je_k\right), \quad i, j, k = 1, \cdots, m. \tag{27}$$

Remark 7. A totally real submanifold which is of maximal dimension is known as the Lagrangian submanifold [24].

Definition 8 (see [25]). A nontotally geodesic Lagrangian submanifold M^m of a complex space form $N^{2m}(4c)$ is called H-umbilical if its second fundamental form satisfies

$$\begin{aligned}&h(e_1, e_i) = \mu Je_n, h(e_i, e_m) = \mu Je_i, \quad i = 1 \cdots m-1,\\ &h(e_m, e_m) = \lambda Je_m, h\left(e_i, e_j\right) = 0, 1 \le i \ne j \le m-1,\end{aligned} \tag{28}$$

for some functions μ and λ with respect to an orthonormal frame $\{e_1, \cdots, e_m\}$, where J is the complex structure of $N^{2m}(4c)$.

Theorem 9. *Let M^m be a totally real submanifold of maximal dimension $m(m \ge 2)$ in a connected complex space form $N(f_1, f_2)$ of dimension $2m$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^m.*

Then, for any unit tangent vector U to M^m

$$\frac{m(m-1)}{4}\|\mathscr{H}\|^2 \geq Ric(U) - f_1(m-1) - [\psi_2 + \psi_1(1-m)]\beta_1(U,U) + \psi_2 trace\beta_1 - \psi_2(\psi_1-\psi_2)[\beta_2(U,U) + trace\beta_2] - (m-1)(\psi_1-\psi_2)\pi(\zeta), \tag{29}$$

and the equality holds in (29) identically if and only if either

(1) M^m is totally geodesic, provided that $m > 2$, or

(2) $m = 2$ and M^2 is a $\mathscr{H}$-umbilical Lagrangian surface with $\lambda = 3\mu$

Proof. As Γ is tangent to M^m, we have

$$\zeta = \tilde{\zeta}, \quad \mathscr{H} = \tilde{\mathscr{H}}. \tag{30}$$

Let us assume an orthonormal basis $\{e_1 = U, e_2, \cdots, e_m\} \subset T_pM^m$ and $\{e_{m+1} = Je_1, \cdots, e_{2m} = Je_m\} \subset T_p^\perp M^m$ at point $p \in M^m$ with unit vector $U \in T_pM^m$. Then, by combining (16) and (19) and substituting $U = W = e_j$ and $V = Z = e_1$, for $j = 2, \cdots, m$, we get

$$\begin{aligned} R(e_j, e_1, e_1, e_j) &= f_1\{g(e_1,e_1)g(e_j,e_j) - g(e_j,e_1)g(e_1,e_j)\} \\ &+ f_2\{g(e_j, Je_1)g(Je_1,e_j) - g(e_1,Je_1)g(Je_j,e_j) \\ &+ 2g(e_j,Je_1)g(Je_1,e_j)\} + \psi_1\beta_1(e_j,e_1)g(e_1,e_j) \\ &- \psi_1\beta_1(e_1,e_1)g(e_j,e_j) + \psi_2\beta_1(e_1,e_j)g(e_j,e_1) \\ &- \psi_2\beta_1(e_j,e_j)g(e_1,e_1) + \psi_2(\psi_1-\psi_2)g(e_j,e_1)\beta_2(e_1,e_j) \\ &- \psi_2(\psi_1-\psi_2)g(e_1,e_1)\beta_2(e_j,e_j) + g(\zeta(e_j,e_j),\zeta(e_1,e_1)) \\ &- g(\zeta(e_1,e_j),\zeta(e_j,e_1)) - (\psi_1-\psi_2)g(\zeta(e_1,e_1),\Gamma^\perp)g(e_j,e_j) \\ &- (\psi_2-\psi_1)g(\zeta(e_j,e_1),\Gamma^\perp)g(e_1,e_j). \end{aligned} \tag{31}$$

Taking the summation over $j = 2, \cdots, m$, we find

$$\begin{aligned} \mathrm{Ric}(U) &= f_1(m-1) + [\psi_2 + \psi_1(1-m)]\beta_1(U,U) - \psi_2\mathrm{trace}\beta_1 \\ &+ \psi_2(\psi_1-\psi_2)[\beta_2(U,U) - \mathrm{trace}\beta_2] \\ &- (m-1)(\psi_1-\psi_2)\pi(\zeta) + \sum_{s=1}^{m}\sum_{j=2}^{m}\left(\zeta_{11}^s\zeta_{jj}^s - \left(\zeta_{1j}^s\right)^2\right), \end{aligned} \tag{32}$$

which implies

$$\begin{aligned} &\mathrm{Ric}(U) - f_1(m-1) - [\psi_2 + \psi_1(1-m)]\beta_1(U,U) + \psi_2\mathrm{trace}\beta_1 \\ &- \psi_2(\psi_1-\psi_2)[\beta_2(U,U) - \mathrm{trace}\beta_2] + (m-1)(\psi_1-\psi_2)\pi(\zeta) \\ &= \sum_{s=1}^{m}\sum_{j=2}^{m}\left(\zeta_{11}^s\zeta_{jj}^s - \left(\zeta_{1j}^s\right)^2\right) \leq \sum_{s=1}^{m}\sum_{j=2}^{m}\zeta_{11}^s\zeta_{jj}^s - \sum_{j=2}^{m}\left(\zeta_{1j}^1\right)^2 - \sum_{j=2}^{m}\left(\zeta_{1j}^j\right)^2. \end{aligned} \tag{33}$$

From the above equation and (26), it is easy to see that

$$\begin{aligned} &\mathrm{Ric}(U) - f_1(m-1) - [\psi_2 + \psi_1(1-m)]\beta_1(U,U) + \psi_2\mathrm{trace}\beta_1 \\ &- \psi_2(\psi_1-\psi_2)[\beta_2(U,U) - \mathrm{trace}\beta_2] + (m-1)(\psi_1-\psi_2)\pi(\zeta) \\ &\leq \sum_{s=1}^{m}\sum_{j=2}^{m}\zeta_{11}^s\zeta_{jj}^s - \sum_{j=2}^{m}\left(\zeta_{11}^j\right)^2 - \sum_{j=2}^{m}\left(\zeta_{jj}^1\right)^2. \end{aligned} \tag{34}$$

Putting

$$\begin{aligned} f_1\left(\zeta_{11}^1, \zeta_{22}^1, \cdots, \zeta_{mm}^1\right) &= \zeta_{11}^1\sum_{j=2}^{m}\zeta_{jj}^1 - \sum_{j=2}^{m}\left(\zeta_{jj}^1\right)^2, \\ f_s(\zeta_{11}^s, \zeta_{22}^s, \cdots, \zeta_{mm}^s) &= \zeta_{11}^s\sum_{j=2}^{m}\zeta_{jj}^s - \left(\zeta_{11}^s\right)^2, \quad \forall s = 2, \cdots, m, \end{aligned} \tag{35}$$

and combining Lemma 5 with the relation $mH^1 = \zeta_{11}^1 + \zeta_{22}^1 + \cdots + \zeta_{mm}^1$, we obtain

$$f_1\left(\zeta_{11}^1, \zeta_{22}^1, \cdots, \zeta_{mm}^1\right) \leq \frac{m-1}{4m}\left(m\mathscr{H}^1\right)^2 = \frac{m(m-1)}{4}\left(\mathscr{H}^1\right)^2. \tag{36}$$

Then, by Lemma 6 for $s = 2, \cdots, m$, we get

$$f_s(\zeta_{11}^s, \zeta_{22}^s, \cdots, \zeta_{mm}^s) \leq \frac{1}{8}(m\mathscr{H}^s)^2 = \frac{m^2}{8}(\mathscr{H}^s)^2 \leq \frac{m(m-1)}{4}(\mathscr{H}^s)^2. \tag{37}$$

We find from (34), (36), and (37) that

$$\begin{aligned} \mathrm{Ric}(U) &\leq f_1(m-1) + [\psi_2 + \psi_1(1-m)]\beta_1(U,U) - \psi_2\mathrm{trace}\beta_1 \\ &+ \psi_2(\psi_1-\psi_2)[\beta_2(U,U) - \mathrm{trace}\beta_2] \\ &- (m-1)(\psi_1-\psi_2)\pi(\zeta), + \frac{m(m-1)}{4}\|\mathscr{H}\|^2, \end{aligned} \tag{38}$$

which is the desired inequality (29).□ □

Now, we discuss the equality cases.

Case 1. For $m > 2$, if $Je_1 \| \mathscr{H}$, then

$$\mathscr{H}^s = 0, \tag{39}$$

for all $s > 1$. Therefore, using Lemma 6, we derive

$$\begin{aligned} \zeta_{1j}^1 = \zeta_{11}^j &= \frac{m\mathscr{H}^j}{4} = 0, \quad \text{for all } j > 1, \\ \zeta_{jk}^1 &= 0, \quad \text{for all } j, k > 1, j \neq k. \end{aligned} \tag{40}$$

Further, Lemma 5 yields

$$\zeta_{11}^{1} = (m+1)\frac{H^1}{2}, \zeta_{jj}^{1} = \frac{H^1}{2}, \quad \text{for all } j > 1. \tag{41}$$

In (33), we see that $\mathrm{Ric}(U) = \mathrm{Ric}(e_1)$. In the same way, by deriving $\mathrm{Ric}(e_2)$ and making use of the equality, we conclude that

$$\zeta_{2j}^{s} = \zeta_{js}^{2} = 0, \quad \text{for all } s \neq 2, j \neq 2, s \neq j. \tag{42}$$

In consequence, we find

$$\frac{\zeta_{11}^{2}}{n+1} = \zeta_{22}^{2} = \cdots = \zeta_{mm}^{2} = \frac{\mathscr{H}^2}{2} = 0. \tag{43}$$

We see that the equality holds for every unit tangent vectors. The above conclusion is also valid for (ζ_{jk}^{s}). Thus,

$$\zeta_{2j}^{2} = \zeta_{22}^{j} = \frac{\mathscr{H}^j}{2} = 0, \quad \forall j \geq 3. \tag{44}$$

Then, the only possible nonzero entries for (ζ_{jk}^{2}) (resp., for (ζ_{jk}^{s})) are

$$\zeta_{12}^{2} = \zeta_{21}^{2} = \zeta_{22}^{1} = \frac{\mathscr{H}^1}{2}\left(\text{respectively } \zeta_{1s}^{s} = \zeta_{s1}^{s} = \zeta_{ss}^{1} = \frac{\mathscr{H}^1}{2}, \forall s \geq 3\right). \tag{45}$$

Substituting $U = Z = e_2$ and $V = W = e_j, j = 2, \cdots, m$ in (16), we derive

$$\tilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \left(\frac{\mathscr{H}^1}{2}\right)^2, \quad \forall j \geq 3. \tag{46}$$

On the other hand, if we substitute $U = Z = e_2$ and $V = W = e_1$ in (16), we get

$$\tilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - (m+1)\left(\frac{\mathscr{H}^1}{2}\right)^2 + \left(\frac{\mathscr{H}^1}{2}\right)^2. \tag{47}$$

Using (46) and (47), we find

$$\begin{aligned} &\mathrm{Ric}(e_2) - f_1(n-1) - [\psi_1 + \psi_2 - n\psi_1]\beta_1(e_2, e_2) + \psi_1 \mathrm{trace}\beta_1 \\ &\quad - \psi_2(\psi_1 - \psi_2)[m\beta_2(e_2, e_2) - \mathrm{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ &= 2(m-1)\left(\frac{\mathscr{H}^1}{2}\right)^2. \end{aligned} \tag{48}$$

Moreover, the equality case of (29) implies that

$$\begin{aligned} &\mathrm{Ric}(e_2) - f_1(m-1) - [\psi_1 + \psi_2 - m\psi_1]\beta_1(U, U) + \psi_1 \mathrm{trace}\beta_1 \\ &\quad - \psi_2(\psi_1 - \psi_2)[m\beta_2(U, U) - \mathrm{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ &= m(m-1)\left(\frac{\mathscr{H}^1}{2}\right)^2. \end{aligned} \tag{49}$$

Using the fact $m \neq 1, 2$, by (48) and (49), it is easy to see that $\mathscr{H}^1 = 0$. This implies that M^m is a totally geodesic in $N^{2m}(4c)$.

Case 2. In case $m = 2$, M^2 is nontotally geodesic, then $\zeta(e_1, e_1) = \lambda e_3, \zeta(e_2, e_2) = \mu e_3, \zeta(e_1, e_2) = \mu e_4$, together with $\lambda = 3\mu$. This proves that M^2 is $\mathscr{H}$-umbilical surface.

The above theorem gives the following results.

Corollary 10. *Let M^m be a totally real submanifold of maximal dimension $m (m \geq 2)$ in a connected complex space form $N(f_1, f_2)$ of dimension 2m with a semisymmetric metric connection such that the vector field Γ is tangent to M^m. Then, for any unit tangent vector U to M^m*

$$Ric(U) \leq \frac{m(m-1)}{4}\|\mathscr{H}\|^2 + f_1(m-1) + (2-m)\beta_1(U, U) - trace\beta_1, \tag{50}$$

and the equality holds in (50) identically if and only if either

(1) M^m is totally geodesic, provided $m > 2$, or

(2) $m = 2$ and M^2 is a $\mathscr{H}$-umbilical Lagrangian surface with $\lambda = 3\mu$

Proof. Using the fact $\psi_1 = \psi_2 = 1$ together with Theorem 9, the result directly follows.□ □

Remark 11. It is worthy to mention here that Corollary 10 together with Remark 4 is the main result for complex case of the paper [26].

Corollary 12. *Let M^m be a totally real submanifold of maximal dimension $m (m \geq 2)$ in a connected complex space form $N(f_1, f_2)$ of dimension 2m with a semisymmetric nonmetric connection such that the vector field Γ is tangent to M^m. Then, for any unit tangent vector U to M^m*

$$Ric(U) \leq \frac{m(m-1)}{4}\|\mathscr{H}\|^2 + f_1(m-1) + (1-m)\beta_1(U, U) + (m-1)\pi(\zeta), \tag{51}$$

and the equality holds in (51) identically if and only if either

(1) M^m is totally geodesic, provided $n > 2$, or

(2) M^2 is a $\mathscr{H}$-umbilical Lagrangian surface with $\lambda = 3\mu$

Proof. Using the fact $\psi_1 = 1$ and $\psi_2 = 0$ together with Theorem 9, the result directly follows.□ □

4. Characterization of Legendrian Submanifold in Generalized Sasakian Space Form

Let a $(2m+1)$-dimensional almost contact metric manifold N^{2m+1} furnished with the almost complex structure (φ, ξ, η, g), where φ is a (1,1) tensor field, ξ is the structure vector field, η, the 1-form, and g is the Riemannian metric on N^{2m+1}. Then, following relations hold good:

$$\varphi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V). \tag{52}$$

It also follows from the above relations that

$$\varphi\xi = 0,\ \eta(\varphi U) = 0,\ \eta(U) = g(U, \xi),\ g(\varphi U, V) + g(U, \varphi V) = 0, \tag{53}$$

for all vector fields U, V on N.

Let $(N, \varphi, \xi, \eta, g)$ be an almost contact metric manifold whose curvature tensor satisfies [27]

$$\begin{aligned}\tilde{\tilde{R}}(U, V)Z = {} & f_1\{g(V, Z)U - g(U, Z)V\} + f_2\{g(U, \varphi Z)\varphi V \\ & - g(V, \varphi Z)\varphi U + 2g(U, \varphi V)\varphi Z\} + f_3\{\eta(U)\eta(Z)V \\ & - \eta(V)\eta(Z)U + g(U, Z)\eta(V)\xi - g(V, Z)\eta(U)\xi\},\end{aligned} \tag{54}$$

for all vector fields U, V, Z on N, where f_1, f_2, f_3 are differentiable functions on N. Then, $N(f_1, f_2, f_3)$ is said to be a generalized Sasakian space form.

Remark 13. The generalized Sasakian space forms are [27]

(1) Sasakian space forms if $f_1 = (c+3)/4, f_2 = f_3 = (c-1)/4$

(2) Kenmotsu space forms if $f_1 = (c-3)/4$ and $f_2 = f_3 = (c+1)/4$

(3) cosymplectic space forms if $f_1 = f_2 = f_3 = c/4$

From (13) and (54), we have

$$\begin{aligned}\bar{R}(U, V)Z = {} & f_1\{g(V, Z)U - g(U, Z)V\} + f_2\{g(U, \varphi Z)\varphi V \\ & - g(V, \varphi Z)\varphi U + 2g(U, \varphi V)\varphi Z\} + f_3\{\eta(U)\eta(Z)V \\ & - \eta(V)\eta(Z)U + g(U, Z)\eta(V)\xi - g(V, Z)\eta(U)\xi\} \\ & + \psi_1\beta_1(U, Z)g(V, W) - \psi_1\beta_1(V, Z)g(U, W) \\ & + \psi_2\beta_1(V, W)g(U, Z) - \psi_2\beta_1(U, W)g(V, Z) \\ & + \psi_2(\psi_1 - \psi_2)g(U, Z)\beta_2(V, W) - \psi_2(\psi_1 - \psi_2)g(V, Z)\beta_2(U, W).\end{aligned} \tag{55}$$

A submanifold M^m of an almost contact manifold N^{2n+1} normal to ξ is called a C-totally real submanifold. On such a submanifold, φ maps any tangent vector to M^m at $p \in M^m$ into the normal space $T_p^{\perp}M^m$. In particular, if $n = m$, i.e., M^m has maximum dimension, then it is a Legendrian submanifold. For a Legendrian submanifold M^m, if $\{e_1, \cdots, e_m\}$ and $\{e_{m+1} = \varphi e_1, \cdots, e_{2m} = \varphi e_m, e_{2m+1} = \xi\}$ be tangent orthonormal frame and normal orthonormal frame, respectively, on M^m. One has

$$\tilde{A}_{JU}V = \tilde{A}_{JV}U, \quad U, V \in T_pM, \tag{56}$$

or equivalently,

$$\tilde{\zeta}_{ij}^k = \tilde{A}_{ik}^j = \tilde{\zeta}_{jk}^i, \quad \forall i, j, k = 1, \cdots, m, \tag{57}$$

where $\tilde{A}^k$ is the shape operator with respect to $\bar{\nabla}$ and

$$\tilde{\zeta}_{ij}^{\ k} = g\left(\tilde{\zeta}\left(e_i, e_j\right), Je_k\right), \quad i, j, k = 1, \cdots, m. \tag{58}$$

Definition 14 (see [28]). A nontotally geodesic Legendrian submanifold M^m of a Sasakian space form $N^{2m+1}(4c)$ is called H-umbilical if its second fundamental form satisfies

$$\begin{aligned} & h(e_1, e_1) = \lambda\phi e_1, h(e_2, e_2) = \cdots = h(e_m, e_m) = \mu\phi e_1, \\ & h\left(e_1, e_j\right) = \mu\phi e_j, h\left(e_j, e_k\right) = 0, \quad j \neq k, j, k = 2 \cdots, m,\end{aligned} \tag{59}$$

for some functions μ and λ with respect to an orthonormal frame $\{e_1, \cdots, e_m\}$, where ϕ is the contact structure of $N^{2m+1}(4c)$.

Theorem 15. *Let M^m be a totally real submanifold of maximal dimension $m(m \geq 2)$ in a generalized Sasakian space form $N(f_1, f_2, f_3)$ of dimension $2m+1$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^m. Then, for any unit tangent vector U to M^m*

$$\begin{aligned}Ric(U) \leq {} & \frac{m(m-1)}{4}\|\mathscr{H}\|^2 + f_1(m-1) + [\psi_2 + \psi_1(1-m)]\beta_1(U, U) \\ & - \psi_2 trace\beta_1 + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) + trace\beta_2] \\ & + (m-1)(\psi_1 - \psi_2)\pi(\zeta)\end{aligned} \tag{60}$$

and the equality holds in (60) identically if and only if either

(1) M^m is totally geodesic, provided $m > 2$, or

(2) $m = 2$ and M^2 is a $\mathscr{H}$-umbilical Legendrian surface with $\lambda = 3\mu$

Proof. As Γ is tangent to M^m, we have

$$\zeta = \tilde{\zeta}, \mathscr{H} = \tilde{\mathscr{H}}. \tag{61}$$

Let us assume an orthonormal basis $\{e_1 = U, e_2, \cdots, e_m\} \subset T_pM^m$ and $\{e_{m+1} = \varphi e_1, \cdots, e_{2m} = \varphi e_m, e_{2m+1} = \xi\} \subset T_p^{\perp}M^m$ at point $p \in M^m$ with unit vector $U \in T_pM^m$. Then, by combining (16) and (55) and substituting $U = W = e_j$ and $V = Z = e_1$ and summing over $j = 2, \cdots, m$, we compute

$$\begin{aligned}\operatorname{Ric}(U) = f_1(m-1) + [\psi_2 + \psi_1(1-m)]\beta_1(U,U) - \psi_2 \operatorname{trace}\beta_1 \\ + \psi_2(\psi_1 - \psi_2)[\beta_2(U,U) - \operatorname{trace}\beta_2] \\ - (m-1)(\psi_1 - \psi_2)\pi(\zeta) + \sum_{s=1}^{m}\sum_{j=2}^{m}\left(\zeta_{11}^s \zeta_{jj}^s - \left(\zeta_{ij}^s\right)^2\right).\end{aligned} \tag{62}$$

From (62) and (57), we deduce

$$\begin{aligned}&\operatorname{Ric}(U) - f_1(m-1) - [\psi_2 + \psi_1(1-m)]\beta_1(U,U) + \psi_2 \operatorname{trace}\beta_1 \\ &\quad - \psi_2(\psi_1 - \psi_2)[\beta_2(U,U) - \operatorname{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ &\leq \sum_{s=1}^{m}\sum_{j=2}^{m}\zeta_{11}^s\zeta_{jj}^s - \sum_{j=2}^{m}\left(\zeta_{11}^j\right)^2 - \sum_{j=2}^{m}\left(\zeta_{jj}^1\right)^2.\end{aligned} \tag{63}$$

Putting

$$\begin{aligned}f_1\left(\zeta_{11}^1, \zeta_{22}^1, \cdots, \zeta_{mm}^1\right) &= \zeta_{11}^1 \sum_{j=2}^{m}\zeta_{jj}^1 - \sum_{j=2}^{m}\left(\zeta_{jj}^1\right)^2, \\ f_s\left(\zeta_{11}^s, \zeta_{22}^s, \cdots, \zeta_{mm}^s\right) &= \zeta_{11}^s \sum_{j=2}^{m}\zeta_{jj}^s - \left(\zeta_{11}^s\right)^2, \quad \forall s = 2, \cdots, m,\end{aligned} \tag{64}$$

and by using the fact $m\mathscr{H}^1 = \zeta_{11}^1 + \zeta_{22}^1 + \cdots + \zeta_{mm}^1$ together with the Lemma 5, we see that

$$f_1\left(\zeta_{11}^1, \zeta_{22}^1, \cdots, \zeta_{mm}^1\right) \leq \frac{m-1}{4m}\left(m\mathscr{H}^1\right)^2 = \frac{m(m-1)}{4}\left(\mathscr{H}^1\right)^2. \tag{65}$$

Application of Lemma 6 for $s = 2, \cdots, m$, gives

$$f_s\left(\zeta_{11}^s, \zeta_{22}^s, \cdots, \zeta_{mm}^s\right) \leq \frac{1}{8}\left(m\mathscr{H}^s\right)^2 = \frac{m^2}{8}\left(\mathscr{H}^s\right)^2 \leq \frac{m(m-1)}{4}\left(\mathscr{H}^s\right)^2. \tag{66}$$

Equations (65), (67), and (68) yield the following relation

$$\begin{aligned}&\operatorname{Ric}(U) - f_1(m-1) - [\psi_2 + \psi_1(1-m)]\beta_1(U,U) + \psi_2 \operatorname{trace}\beta_1 \\ &\quad - \psi_2(\psi_1 - \psi_2)[\beta_2(U,U) - \operatorname{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ &\leq \frac{m(m-1)}{4}\sum_{s=1}^{m}\left(\mathscr{H}^s\right)^2 = \frac{m(m-1)}{4}\|\mathscr{H}\|^2.\end{aligned} \tag{67}$$

Thus, we derive

$$\begin{aligned}\operatorname{Ric}(U) \leq f_1(m-1) + [\psi_2 + \psi_1(1-m)]\beta_1(U,U) - \psi_2 \operatorname{trace}\beta_1 \\ + \psi_2(\psi_1 - \psi_2)[\beta_2(U,U) - \operatorname{trace}\beta_2] \\ + (m-1)(\psi_1 - \psi_2)\pi(\zeta), + \frac{m(m-1)}{4}\|\mathscr{H}\|^2,\end{aligned} \tag{68}$$

which is the desired inequality (60).□ □

Now, we discuss the equality cases.

Case 1. For $m > 2$, if $Je_1 \| \mathscr{H}$. Then,

$$\mathscr{H}^s = 0, \tag{69}$$

for all $s > 1$. Therefore, using Lemma 6, we derive

$$\zeta_{1j}^1 = \zeta_{11}^j = \frac{m\mathscr{H}^j}{4} = 0, \quad \text{for all } j > 1, \tag{70}$$

$$\zeta_{jk}^1 = 0, \quad \text{for all } j, k > 1,\ j \neq k. \tag{71}$$

Further, Lemma 5 yields

$$\zeta_{11}^1 = (m+1)\frac{H^1}{2},\ \zeta_{jj}^1 = \frac{H^1}{2}, \quad \text{for all } j > 1. \tag{72}$$

In (63), we see that $\operatorname{Ric}(U) = \operatorname{Ric}(e_1)$. In the same way, by deriving $\operatorname{Ric}(e_2)$ and making use of the equality, we conclude that

$$\zeta_{2j}^s = \zeta_{js}^2 = 0, \quad \text{for all } s \neq 2, j \neq 2, s \neq j. \tag{73}$$

In consequence, we find

$$\frac{\zeta_{11}^2}{m+1} = \zeta_{22}^2 = \cdots = \zeta_{mm}^2 = \frac{\mathscr{H}^2}{2} = 0. \tag{74}$$

We see that the equality holds for every unit tangent vectors. The above conclusion is also valid for (ζ_{jk}^s). Thus,

$$\zeta_{2j}^2 = \zeta_{22}^j = \frac{\mathscr{H}^j}{2} = 0, \quad \forall j \geq 3. \tag{75}$$

Then, the only possible nonzero entries for (ζ_{jk}^2) (resp., for (ζ_{jk}^s)) are

$$\zeta_{12}^2 = \zeta_{21}^2 = \zeta_{22}^1 = \frac{\mathscr{H}^1}{2}\left(\text{resp.} \zeta_{1s}^s = \zeta_{s1}^s = \zeta_{ss}^1 = \frac{\mathscr{H}^1}{2}, \forall s \geq 3\right). \tag{76}$$

Substituting $U = Z = e_2$ and $V = W = e_j, j = 2, \cdots, m$ in

(16), we obtain

$$\tilde{R}\left(e_2, e_j, e_2, e_j\right) = R\left(e_2, e_j, e_2, e_j\right) - \left(\frac{\mathscr{H}^1}{2}\right)^2, \quad \forall j \geq 3. \quad (77)$$

On the other hand, if we put $U = Z = e_2$ and $V = W = e_1$ in (16), we get

$$\tilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - (m+1)\left(\frac{\mathscr{H}^1}{2}\right)^2 + \left(\frac{\mathscr{H}^1}{2}\right)^2. \quad (78)$$

From (77) and (78), it follows that

$$\begin{aligned} &\mathrm{Ric}(e_2) - f_1(m-1) - [\psi_1 + \psi_2 - m\psi_1]\beta_1(e_2, e_2) + \psi_1 \mathrm{trace}\beta_1 \\ &\quad - \psi_2(\psi_1 - \psi_2)[m\beta_2(e_2, e_2) - \mathrm{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ &= 2(m-1)\left(\frac{\mathscr{H}^1}{2}\right)^2. \end{aligned} \quad (79)$$

Moreover, using the equality case of (29), we see that

$$\begin{aligned} &\mathrm{Ric}(e_2) - f_1(m-1) - [\psi_1 + \psi_2 - m\psi_1]\beta_1(U, U) + \psi_1 \mathrm{trace}\beta_1 \\ &\quad - \psi_2(\psi_1 - \psi_2)[m\beta_2(U, U) - \mathrm{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ &= m(m-1)\left(\frac{\mathscr{H}^1}{2}\right)^2. \end{aligned} \quad (80)$$

Indeed $m \neq 1, 2$, with (81) and (84), we find $\mathscr{H}^1 = 0$. This implies that M^m is a totally geodesic in $N^{2m+1}(c)$.

Case 2. In the case $m = 2$, M^2 is nontotally geodesic, then $\zeta(e_1, e_1) = \lambda e_3, \zeta(e_2, e_2) = \mu e_3, \zeta(e_1, e_2) = \mu e_4$ together with $\lambda = 3\mu$. This proves that M^2 is $\mathscr{H}$-umbilical surface.

Remark 16. If we consider the structure vector field ξ tangential to the submanifold M, then we have the following result.

Theorem 17. *Let M^{m+1} be a totally real submanifold of maximal dimension $m+1 (m \geq 2)$ in a generalized Sasakian space form $N(f_1, f_2, f_3)$ of dimension $2m+1$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^{m+1}. Then, for any unit tangent vector U to M^{m+1},*

$$\begin{aligned} Ric(U) \leq &\frac{m(m+1)}{4}\|\mathscr{H}\|^2 + f_1 m - \left\{(m-1)\eta^2(e_1) + 1\right\} f_3 \\ &+ [\psi_2 + \psi_1 m]\beta_1(U, U) \\ &- \psi_2 trace\beta_1 + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) + trace\beta_2] \\ &+ m(\psi_1 - \psi_2)\pi(\zeta), \end{aligned} \quad (81)$$

and the equality holds in (81) identically if and only if either

(1) M^{m+1} is totally geodesic, provided $m > 2$, or

(2) $m = 2$ and M^2 is a $\mathscr{H}$-umbilical Legendrian surface with $\lambda = 3\mu$

Proof. We obtain the proof on the same lines of the proof for Theorem 15 additionally assuming an orthonormal basis

$$\{e_1 = U, e_2, \cdots, e_m, e_{m+1}, e_{m+2}, \cdots, e_{2m+1}\}, \quad (82)$$

such that $e_1, e_2, \cdots, e_m, e_{m+1} \in T_pM$.

As a consequence of Theorem 15, we obtain the following results.□ □

Corollary 18. *Let M^m be a totally real submanifold of maximal dimension $m (m \geq 2)$ in a Sasakian space form $N(c)$ of dimension $2m+1$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^m. Then, for any unit tangent vector U to M^m*

$$\begin{aligned} Ric(U) \leq &\frac{m(m-1)}{4}\|\mathscr{H}\|^2 + \frac{c+3}{4}(m-1) \\ &+ [\psi_2 + \psi_1(1-m)]\beta_1(U, U) - \psi_2 trace\beta_1 \\ &+ \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) + trace\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta), \end{aligned} \quad (83)$$

and the equality holds in (83) identically if and only if either

(1) M^m is totally geodesic, provided $m > 2$, or

(2) $m = 2$ and M^2 is a $\mathscr{H}$-umbilical Legendrian surface with $\lambda = 3\mu$

Proof. The proof follows immediately from Theorem 15 by putting $f_1 = (c+3)/4, f_2 = f_3 = (c-1)/4$.□ □

Corollary 19. *Let M^m be a totally real submanifold of maximal dimension $m (m \geq 2)$ in a Kenmotsu space form $N(c)$ of dimension $2m+1$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^m. Then, for any unit tangent vector U to M^m*

$$\begin{aligned} Ric(U) \leq &\frac{m(m-1)}{4}\|\mathscr{H}\|^2 + \frac{c-3}{4}(m-1) \\ &+ [\psi_2 + \psi_1(1-m)]\beta_1(U, U) - \psi_2 trace\beta_1 \\ &+ \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) + trace\beta_2] \\ &+ (m-1)(\psi_1 - \psi_2)\pi(\zeta), \end{aligned} \quad (84)$$

and the equality holds in (84) identically if and only if either

(1) M^m is totally geodesic, provided $m > 2$, or

(2) $m = 2$ and M^2 is a $\mathscr{H}$-umbilical Legendrian surface with $\lambda = 3\mu$

Proof. The proof follows immediately from Theorem 15 by replacing $f_1 = (c-3)/4$ and $f_2 = f_3 = (c+1)/4$.□ □

Corollary 20. *Let M^m be a totally real submanifold of maximal dimension $m (m \geq 2)$ in a cosymplectic space form $N(c)$ of dimension $2m+1$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^m. Then, for any unit tangent vector U to M^m*

$$\begin{aligned} Ric(U) \leq{} & \frac{m(m-1)}{4} \|\mathscr{H}\|^2 + \frac{c-3}{4}(m-1) \\ & + [\psi_2 + \psi_1(1-m)]\beta_1(U,U) - \psi_2 trace\beta_1 \\ & + \psi_2(\psi_1 - \psi_2)[\beta_2(U,U) + trace\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta), \end{aligned} \tag{85}$$

and the equality holds in (85) identically if and only if either

(1) M^m is totally geodesic, provided $m > 2$, or

(2) $m = 2$ and M^2 is a $\mathscr{H}$-umbilical Legendrian surface with $\lambda = 3\mu$

Proof. The proof follows immediately from Theorem 15 by substituting $f_1 = f_2 = f_3 = c/4$.□ □

Corollary 21. *Let M^m be a totally real submanifold of maximal dimension $m (m \geq 2)$ in a generalized Sasakian space form $N(f_1, f_2, f_3)$ of dimension $2m+1$ with a semisymmetric metric connection such that the vector field Γ is tangent to M^m. Then, for any unit tangent vector U to M^m*

$$Ric(U) \leq \frac{m(m-1)}{4} \|\mathscr{H}\|^2 + f_1(m-1) + (2-m)\beta_1(U,U) - trace\beta_1, \tag{86}$$

and the equality in (60) holds identically if and only if either

(1) M^m is totally geodesic, provided $m > 2$, or

(2) $m = 2$ and M^2 is a $\mathscr{H}$-umbilical Legendrian surface with $\lambda = 3\mu$

Proof. Using the fact that $\psi_1 = \psi_2 = 1$ together with Theorem 15, the result directly follows.□ □

Remark 22. It is worthy to mention here that Corollary 21 together with Remark 13 (1) is the main result of the paper [6].

Corollary 23. *Let M^m be a totally real submanifold of maximal dimension $m (m \geq 2)$ in a generalized Sasakian space form $N(f_1, f_2, f_3)$ of dimension $2m+1$ with a semisymmetric nonmetric connection such that the vector field Γ is tangent to M^m. Then, for any unit tangent vector U to M^m*

$$Ric(U) \leq \frac{m(m-1)}{4} \|\mathscr{H}\|^2 + f_1(m-1) + (1-m)\beta_1(U,U) + (m-1)\pi(\zeta), \tag{87}$$

and the equality in (87) holds identically if and only if either

(1) M^m is totally geodesic, provided $m > 2$, or

(2) $m = 2$ and M^2 is a $\mathscr{H}$-umbilical Legendrian surface with $\lambda = 3\mu$

Proof. Using the fact that $\psi_1 = 1$ and $\psi_2 = 0$ together with Theorem 15, the result directly follows.□ □

Remark 24. All the above cases for Theorem 15 can be seen in the case of Theorem 17 as well.

Remark 25. Examples of totally geodesic submanifolds, H-umbilical Lagrangian submanifolds, and H-umbilical Legendrian submanifolds, i.e., examples of submanifolds attaining the equality case of the inequalities stated in this article, can be found in [25, 29, 30].

Data Availability

No data is used for the research

Conflicts of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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